

THE TORSION IN SYMMETRIC POWERS ON CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

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ABSTRACT. In this paper we prove that for a fixed neat principal congruence subgroup of a Bianchi group the order of the torsion part of its second cohomology group with coefficients in an integral lattice associated to the m -th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ grows exponentially in m^2 . We give upper and lower bounds for the growth rate. Our result extends a result of W. Müller and S. Marshall, who proved the corresponding statement for closed arithmetic 3-manifolds, to the finite-volume case. We also prove a limit multiplicity formula for twisted combinatorial Reidemeister torsion on higher dimensional hyperbolic manifolds.

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1. INTRODUCTION

The torsion in the cohomology of arithmetic groups has recently attracted new interest from number theorists. Without aiming at completeness, we refer for example to [BV13], [CV12], [Eme14] and [Sch13]. In this paper, we study the twisted cohomological torsion quantitatively for a fixed principal congruence subgroup of a Bianchi group under a variation of the local system. Bianchi groups represent all classes of non-uniform lattices in $\mathrm{SL}_2(\mathbb{C})$; thus our result complements the study of this question for arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ defined over imaginary quadratic fields done by Simon Marshall and Werner Müller in [MM13] (where the authors give an equality for the asymptotic torsion size, while we only get upper and lower bounds for the growth rate).

To state our main result more precisely, we need to introduce some notation. Let $D \in \mathbb{N}$ be square-free and let $F = \mathbb{Q}(\sqrt{-D})$ be the associated imaginary quadratic number field

with ring of integers \mathcal{O}_D . Let $\Gamma_D := \mathrm{SL}_2(\mathcal{O}_D)$. Then Γ_D is an arithmetic subgroup of $\mathrm{SL}_2(\mathbb{C})$ which acts on $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2) \cong \mathbb{H}^3$ and the quotient $\Gamma_D \backslash \mathbb{H}^3$ is a hyperbolic orbifold of finite volume. If \mathfrak{a} is a non-zero ideal of \mathcal{O}_D , we let $\Gamma(\mathfrak{a})$ denote the principal congruence subgroup of level \mathfrak{a} . This is a finite-index subgroup of Γ_D which is neat (i.e. none of its non-unipotent elements have a root of unity as an eigenvalue, in particular it is torsion-free) as soon as the norm $N(\mathfrak{a})$ is sufficiently large ($N(\mathfrak{a}) \geq 9$ suffices). We shall assume this from now on. Thus, $X_{\mathfrak{a}} := \Gamma(\mathfrak{a}) \backslash \mathbb{H}^3$ is an arithmetic hyperbolic manifold of finite volume. It is never compact and has finitely many cusps, whose number we shall denote by $\kappa(\Gamma(\mathfrak{a}))$. For $m \in \mathbb{N}$ let ρ_m be the natural representation of $\mathrm{SL}_2(\mathbb{C})$ on the m th symmetric power $V(m) := \mathrm{Sym}^m \mathbb{C}^2$ of its standard representation on \mathbb{C}^2 . Then there exists a \mathbb{Z} -lattice $\Lambda(m)$ in $V(m)$ which is preserved by $\rho_m(\Gamma_D)$ (one can simply take $\Lambda(m) = \mathrm{Sym}^m \mathcal{O}_D^2$).

Now one can form the integral cohomology groups $H^*(\Gamma(\mathfrak{a}); \Lambda(m))$ of the $\Gamma(\mathfrak{a})$ -modules $\Lambda(m)$. These are finitely generated abelian groups and thus they split as

$$H^*(\Gamma(\mathfrak{a}); \Lambda(m)) = H^*(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{free}} \oplus H^*(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{tors}},$$

where the first group in this decomposition is a free finite-rank \mathbb{Z} -module and the second group is a finite abelian group. Moreover, $H^*(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{free}}$ is a lattice in the real cohomology $H^1(\Gamma(\mathfrak{a}), V(m))$. In this paper we are interested in the behaviour of the size of the cohomology group $H^*(\Gamma(\mathfrak{a}); \Lambda(m))$ as m goes to infinity. First we note that

$$(1.1) \quad \dim H^1(\Gamma(\mathfrak{a}), V(m)) = \dim H^2(\Gamma(\mathfrak{a}), V(m)) = \kappa(\Gamma(\mathfrak{a}))$$

for each $m \in \mathbb{N}$. This is easy to verify, see for example section 4. On the other hand, we will show that in degree 2 the size of the torsion part grows exponentially in m^2 as $m \rightarrow \infty$ and we will specify the growth rate. More precisely, the main result of this paper is the following theorem (we stated it for the lattices $\Lambda(m)$ but we prove that the growth rates are the same for any sequence of $\Gamma(\mathfrak{a})$ -invariant lattices in $V(m)$ —see Proposition 7.3).

Theorem A. *There exist constants $C_1(\Gamma_D) > 0$, $C_2(\Gamma_D) > 0$, which depend only on Γ_D such that for each non-zero ideal \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) > C_1(\Gamma_D)$ one has*

$$(1.2) \quad \liminf_{m \rightarrow \infty} \frac{\log |H^2(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{tors}}|}{m^2} \geq \frac{1}{2} \cdot \frac{\mathrm{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{C_1(\Gamma_D)}{N(\mathfrak{a})} \right) > 0$$

and

$$(1.3) \quad \limsup_{m \rightarrow \infty} \frac{\log |H^2(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{tors}}|}{m^2} \leq \frac{\mathrm{vol}(X_{\mathfrak{a}})}{\pi} \left(1 + \frac{C_2(\Gamma_D)}{N(\mathfrak{a})} \right).$$

Finally, we also have

$$(1.4) \quad |H^1(\Gamma(\mathfrak{a}); \Lambda(m))_{\mathrm{tors}}| = O(m \log m),$$

as $m \rightarrow \infty$.

If the class number of F is one and $\mathcal{O}_D^* = \{\pm 1\}$, we can take $C_1(\Gamma_D) = 4$ (this is valid only for $D = -7, -11, -19, -43, -67$ and -163 by the Stark–Heegner Theorem). We shall now briefly sketch our method to prove our main result. The main point in

our case is to establish the lower bound on the torsion given in (1.2), i.e. to establish its exponential growth in m^2 . Let us point out that there are two severe difficulties in the present non-compact case which are not present in the case of compact arithmetic 3-manifolds mentioned above. Firstly, the use of analytic torsion as a main tool is more complicated. Secondly, the real cohomology $H^*(\Gamma, V(m))$ does not vanish in our situation. We shall now describe these issues in more detail.

As already observed by Nicolas Bergeron and Akshay Venkatesh in [BV13], the size of cohomological torsion is closely related to the Reidemeister torsion of the underlying manifold with coefficients in the underlying local system. In the present finite volume case, one has to work with the twisted Reidemeister torsion of the Borel–Serre compactification \overline{X} of X . For technical reasons, in most of the paper we also symmetrize the lattice $\Lambda(m)$ to a lattice $\bar{\Lambda}(m)$ in $\bar{V}(m) := V(m) \oplus V(m)^*$ which is self-dual over \mathbb{Z} (it is then not hard to deduce the estimates in Theorem A from their analogues for $\bar{\Lambda}(m)$ -coefficients). The Reidemeister torsion is then defined with respect to a canonical basis in the cohomology $H^*(X; \bar{V}(m))$ using Eisenstein cohomology classes following Günter Harder [Har75]. By a gluing formula for the Reidemeister torsion, which was obtained by the first author in [Pfa13] building on work of Matthias Lesch [Les13], the Reidemeister torsion can be compared to the regularized analytic torsion of the manifold X . The asymptotic behaviour of the regularized analytic torsion in the finite volume case for a variation of the local system has already been studied by Müller and the first author in [MP12] using the Selberg trace formula. Thus we have to study the error term which occurred in [Pfa13] in the comparison formula between analytic and Reidemeister torsion. It turns out that our study of the error term can be performed without any changes also in the higher dimensional situations. While we do not compute the error term explicitly, we bring it in a form which is sufficient for the application to cohomological torsion. The main point is that the error term depends only on the geometry of the cusps which is very restricted on such manifolds. Also, along the line we can establish limit multiplicity formulae for twisted Reidemeister torsion in the spirit of [BV13] on arithmetic hyperbolic manifolds of finite volume of arbitrary dimension, see Corollary 2.4.

Now we turn to the second aforementioned difficulty. Since the real cohomology of $\Gamma(\mathfrak{a})$ with coefficients in $\bar{V}(m)$ does not vanish in the finite-volume case, we also have to study certain volume factors occurring in the comparison formula between Reidemeister torsion and the size of cohomological torsion. Since our basis in the real cohomology is given by Eisenstein series, this leads to the question about the integrality of certain quotients of L -functions evaluated at positive integers. In the 3-dimensional case, these are Hecke L -functions and we can use the work of R. Damerell [Dam70], [Dam71] to control these quotients. At the moment, we do not know how to do this in higher dimensions.

Concerning the other statements of our main theorem, at least with a worse constant, the upper bound (1.3) can be established in an elementary and completely combinatorial way, without referring to analytic or Reidemeister torsion, see Proposition 8.3. In fact, our approach for the upper bound on the combinatorial torsion generalizes easily to arbitrary dimensions and to arbitrary rays in the weight lattice; it is similar to that used by V. Emery [Eme14] or R. Sauer [Sau14]. We note that if we were able to obtain a proper limit

for the Redemeister torsion instead of the quotient of two such, we would obtain an optimal upper bound for the exponential growth rate of the torsion in the second cohomology, by an argument similar to that used in the proof of the easy part of [Rai13, Lemma 6.14]. The last estimate (1.4) in our theorem is the easiest one to prove, and does not require analytic torsion or any sophisticated tool. Since we work with a \mathbb{Q} -split group the representations are easier to analyze than in the nonsplit case which was dealt with in [MM13, section 4]—we can work globally from the beginning.

We finally remark that Theorem A also holds with the same proof for slightly more general rays of local systems. Namely, the finite dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ are parametrized as $\mathrm{Symm}^{n_1} \otimes \overline{\mathrm{Symm}}^{n_2}$, where $n_1, n_2 \in \mathbb{N}$ and where $\overline{\mathrm{Symm}}^{n_2}$ is the complex conjugate of Symm^{n_2} . Each such representation space carries a canonical \mathbb{Z} -lattice preserved by the action of Γ_D . If we fix n_1 and n_2 with $n_1 \neq n_2$ and let $\rho_m(n_1, n_2)$ be the representation $\mathrm{Symm}^{mn_1} \otimes \overline{\mathrm{Symm}}^{mn_2}$, then the analog of Theorem A holds if we replace the factor m^2 by $m \dim \rho_m(n_1, n_2)$ which grows as m^3 if both n_1 and n_2 are not zero. However, we can by no means remove the assumption $n_1 \neq n_2$. In other words, the ray $\rho_m(1, 1) = \mathrm{Symm}^m \otimes \overline{\mathrm{Symm}}^m$, which is the ray carrying cuspidal cohomology, cannot be studied by our methods. For a fixed $\Gamma(\mathfrak{a})$, we can only show that the size of cohomological torsion with coefficients in the canonical lattice associated to $\rho_m(1, 1,)$ grows at most as $m \mathrm{rk} \rho_m(1, 1) = O(m^3)$, but we can say nothing about the existence of torsion along this ray, i.e. we cannot establish any bound from below. The reason is that here 0 belongs to the essential spectrum of the twisted Laplacian in the middle dimension. Therefore, essentially none of the results on analytic torsion we use in our proof is currently available for $\rho_m(1, 1)$ and also the regulator would be more complicated. We remark that, as far as we know, even in the compact case no result for the growth of torsion along this particular ray has been obtained. For an investigation of the dimension of the (cuspidal) cohomology along this ray, we refer to [FGT10].

This paper is organized as follows : in section 2 we introduce the analytic torsion for cusped manifolds, then in section 3 we study it further for congruence subgroups of Bianchi groups. We introduce the combinatorial (Reidemeister) torsion in section 4 and recall the Cheeger–Müller equality proven in [Pfa13] there. Then we study the intertwining operators in section 5 and 6, first computing them using adeles and then bounding their denominators. The last sections contain the proof of the main theorem: in section 7 we combine the results of the previous sections to prove (1.2), and we prove (1.3) in section 8.

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2. THE REGULARIZED ANALYTIC TORSION FOR COVERINGS

In this section we shall review the definition of regularized traces and the regularized analytic torsion of hyperbolic manifolds X of finite volume. These objects are defined in terms of a fixed choice of truncation parameters on X and there are two different ways to perform such a truncation which are relevant in the present paper. Firstly, one can define

a truncation of X via a fixed choice of Γ -cuspidal parabolic subgroups of G . Secondly, if X is a finite covering of a hyperbolic orbifold X_0 , then a choice of truncation parameters on X_0 gives a truncation on X in terms of which one can define another regularized analytic torsion. We shall compute the difference between the associate regularized analytic torsions explicitly. For more details we refer to [MP12], [MP14a].

We denote by $\mathrm{SO}^0(d, 1)$ the identity-component of the isometry group of the standard quadratic form of signature $(d, 1)$ on \mathbb{R}^{d+1} . Let $\mathrm{Spin}(d, 1)$ denote the universal covering of $\mathrm{SO}^0(d, 1)$. We let either $G := \mathrm{SO}^0(d, 1)$ or $G := \mathrm{Spin}(d, 1)$. We assume that d is odd and write $d = 2n + 1$. Let $K := \mathrm{SO}(d)$, if $G = \mathrm{SO}^0(d, 1)$ or $K := \mathrm{Spin}(d)$, if $G = \mathrm{Spin}(d, 1)$. We let \mathfrak{g} be the Lie algebra of G . Let θ denote the standard Cartan involution of \mathfrak{g} and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the associated Cartan decomposition of \mathfrak{g} , where \mathfrak{k} is the Lie algebra of K . Let B be the Killing form. Then

$$(2.1) \quad \langle X, Y \rangle := -\frac{1}{2(d-1)} B(X, \theta Y)$$

is an inner product on \mathfrak{g} . Moreover, the globally symmetric space G/K , equipped with the G -invariant metric induced by the restriction of (2.1) to \mathfrak{p} is isometric to the d -dimensional real hyperbolic space \mathbb{H}^d . Let $\Gamma \subset G$ be a discrete, torsion-free subgroup. Then $X := \Gamma \backslash \mathbb{H}^d$, equipped with the push-down of the metric on \mathbb{H}^d , is a d -dimensional hyperbolic manifold. We let P be a fixed parabolic subgroup of G with Langlands decomposition $P = M_P A_P N_P$ as in [MP12]. Let \mathfrak{a} denote the Lie algebra of A_P and $\exp : \mathfrak{a} \rightarrow A_P$ the exponential map. Then we fix a restricted root e_1 of \mathfrak{a} in \mathfrak{g} , let $H_1 \in \mathfrak{a}$ be such that $e_1(H_1) = 1$ and define $\iota_P : (0, \infty) \rightarrow A_P$ by $\iota_P(t) := \exp(\log t H_1)$. If P_1 is another parabolic subgroup of G , we fix $k_P \in K$ with $P_1 = k_{P_1} P k_{P_1}^{-1}$ and define $A_{P_1} := k_{P_1} A_P k_{P_1}^{-1}$, $M_{P_1} := k_{P_1} M_P k_{P_1}^{-1}$, $N_{P_1} := k_{P_1} N_P k_{P_1}^{-1}$. Moreover, for $t \in (0, \infty)$ we define $\iota_{P_1}(t) := k_{P_1} \iota_P(t) k_{P_1}^{-1} \in A_{P_1}$. For $Y > 0$ we let $A_{P_1}[Y] := \iota_{P_1}([Y, \infty))$.

A parabolic subgroup P_1 of G is called Γ -cuspidal if $\Gamma \cap N_{P_1}$ is a lattice in N_{P_1} . From now on, we assume that $\mathrm{vol}(X)$ is finite and that Γ is neat in the sense of Borel, i.e. that $\Gamma \cap P_1 = \Gamma \cap N_{P_1}$ for each Γ -cuspidal P_1 . If P_1 is Γ -cuspidal, then for $Y > 0$ we put

$$F_{P_1; \Gamma}(Y) := (\Gamma \cap N_{P_1}) \backslash N_{P_1} \times A_{P_1}[Y] \cong (\Gamma \cap N_{P_1}) \backslash N_{P_1} \times [Y, \infty).$$

We equip $F_{P_1; \Gamma}(Y)$ with the metric $y^{-2} g_{N_{P_1}} + y^{-2} dy^2$ where $g_{N_{P_1}}$ is the push-down of the invariant metric on N_{P_1} induced by the inner product (2.1) restricted to \mathfrak{n}_{P_1} .

Let $\mathrm{Rep}(G)$ denote the set of finite-dimensional irreducible representations of G . For $\rho \in \mathrm{Rep}(G)$ the associated vector space V_ρ possesses a distinguished inner product $\langle \cdot, \cdot \rangle_\rho$ which is called admissible and which is unique up to scaling. We shall fix an admissible inner product on each V_ρ . If $\rho \in \mathrm{Rep}(G)$, then the restriction of ρ to Γ induces a flat vector bundle $E_\rho := \tilde{X} \times_{\rho|_\Gamma} V_\rho$. This bundle is canonically isomorphic to the locally homogeneous bundle $E'_\rho := \Gamma \backslash G \times_{\rho|_K} V_\rho$ induced by the restriction of ρ to K . In particular, since $\rho|_K$ is a unitary representation on $(V_\rho, \langle \cdot, \cdot \rangle_\rho)$, the inner product $\langle \cdot, \cdot \rangle_\rho$ induces a smooth bundle metric on E'_ρ and therefore on E_ρ . For $p = 0, \dots, d$ let $\Delta_p(\rho)$ denote the flat Hodge Laplacian acting on the smooth E_ρ -valued p -forms of X . Since X is complete, $\Delta_p(\rho)$ with domain the smooth, compactly supported E_ρ -valued p -forms is essentially selfadjoint and

its L^2 -closure shall be denoted by the same symbol. Let $e^{-t\Delta_p(\rho)}$ denote the heat semigroup of $\Delta_p(\rho)$ and let

$$K_X^{\rho,p}(t, x, y) \in C^\infty(X \times X; E_\rho \boxtimes E_\rho^*)$$

be the integral kernel of $e^{-t\Delta_p(\rho)}$.

We let \mathfrak{P}_Γ be a fixed set of Γ -cuspidal parabolic subgroups of G . Then \mathfrak{P}_Γ is non-empty if and only if X is non-compact. Moreover, $\kappa(\Gamma) := \#\mathfrak{P}_\Gamma$ equals the number of cusps of X which from now on we assume to be nonzero. The choice of \mathfrak{P}_Γ and of a fixed base-point in \tilde{X} determine an exhaustion of X by smooth compact manifolds $X_{\mathfrak{P}_\Gamma}(Y)$ with boundary, $Y \gg 0$. This exhaustion depends on the choice of \mathfrak{P}_Γ . Then one can show that the integral of $K_X^{\rho,p}(t, x, x)$ over $X_{\mathfrak{P}_\Gamma}(Y)$ has an asymptotic expansion

$$(2.2) \quad \int_{X_{\mathfrak{P}_\Gamma}(Y)} \text{Tr } K_X^{\rho,p}(t, x, x) dx = \alpha_{-1}(t) \log Y + \alpha_0(t) + o(1),$$

as $Y \rightarrow \infty$, [MP12][section 5]. Now one can define the regularized trace $\text{Tr}_{\text{reg}; X; \mathfrak{P}_\Gamma} e^{-t\Delta_p(\rho)}$ of $e^{-t\Delta_p(\rho)}$ with respect to the choice of \mathfrak{P}_Γ by $\text{Tr}_{\text{reg}; X; \mathfrak{P}_\Gamma} e^{-t\Delta_p(\rho)} := \alpha_0(t)$, where $\alpha_0(t)$ is the constant term in the asymptotic expansion in (2.2).

From now on, we also assume that there is a hyperbolic orbifold $X_0 := \Gamma_0 \backslash \mathbb{H}^d$ such that X is a finite covering of X_0 . Let $\pi : X \rightarrow X_0$ denote the covering map. Then if a set of truncation parameters on X_0 , or in other words a set \mathfrak{P}_{Γ_0} of representatives of Γ_0 -cuspidal parabolic subgroups are fixed, one obtains truncation parameters on X by pulling back the truncation on X_0 via π . One can again show that there is an asymptotic expansion

$$(2.3) \quad \int_{\pi^{-1}X_{\mathfrak{P}_{\Gamma_0}}(Y)} \text{Tr } K_X^{\rho,p}(t, x, x) dx = \tilde{\alpha}_{-1}(t) \log Y + \tilde{\alpha}_0(t) + o(1),$$

as $Y \rightarrow \infty$ and one can define the regularized trace with respect to the truncation parameters on X_0 as $\text{Tr}_{\text{reg}; X; X_0} e^{-t\Delta_p(\rho)} := \tilde{\alpha}_0(t)$. This regularized trace depends only on the choice of a set \mathfrak{P}_{Γ_0} of representatives of Γ_0 -cuspidal parabolic subgroups of G . Put

$$(2.4) \quad K_{X; \mathfrak{P}_\Gamma}(t, \rho) := \sum_p (-1)^p p \text{Tr}_{\text{reg}; \mathfrak{P}_\Gamma} e^{-t\Delta_p(\rho)}; \quad K_{X; X_0}(t, \rho) := \sum_p (-1)^p p \text{Tr}_{\text{reg}; X; X_0} e^{-t\Delta_p(\rho)}.$$

Now assume that ρ satisfies $\rho \neq \rho_\theta$. Then one defines the analytic torsion with respect to the two truncations of X by

$$(2.5) \quad \log T_{X; \mathfrak{P}_\Gamma}(\rho) := \left. \frac{1}{2} \frac{d}{ds} \right|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K_{X; \mathfrak{P}_\Gamma}(t, \rho) dt \right);$$

$$(2.6) \quad \log T_{X; X_0}(\rho) := \left. \frac{1}{2} \frac{d}{ds} \right|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K_{X; X_0}(t, \rho) dt \right).$$

Here the integrals converge absolutely and locally uniformly for $\Re(s) > d/2$ and are defined near $s = 0$ by analytic continuation, [MP12, section 7], [MP14a, section 9].

To compare the two analytic torsions in (2.5) and (2.6), we need to introduce some more notation. We fix $\mathfrak{P}_{\Gamma_0} = \{P_{0,1}, \dots, P_{0,\kappa(\Gamma_0)}\}$ and $\mathfrak{P}_\Gamma = \{P_1, \dots, P_{\kappa(\Gamma)}\}$. Then for each $P_j \in$

\mathfrak{P}_Γ there exists a unique $l(j) \in \{1, \dots, \kappa(\Gamma_0)\}$ and a $\gamma_j \in \Gamma_0$ such that $\gamma_j P_j \gamma_j^{-1} = P_{0,l(j)}$. Write

$$(2.7) \quad \gamma_j = n_{0,l(j)} \iota_{P_{0,l(j)}}(t_{P_j}) k_{0,l(j)},$$

$n_{0,l(j)} \in N_{P_{0,l(j)}}$, $t_{P_j} \in (0, \infty)$, $\iota_{P_{0,l(j)}}(t_{P_j}) \in A_{P_{0,l(j)}}$ as above, and $k_{0,l(j)} \in K$. Since $P_{0,l(j)}$ equals its normalizer, the projection of γ_j to $(\Gamma_0 \cap P_{0,l(j)}) \backslash \Gamma_0$ is unique. Moreover, since $P_{0,l(j)}$ is Γ_0 -cuspidal, one has $\Gamma_0 \cap P_{0,l(j)} = \Gamma_0 \cap N_{P_{0,l(j)}} M_{P_{0,l(j)}}$. Thus t_{P_j} depends only on \mathfrak{P}_{Γ_0} and P_j .

Now the analytic torsions $\log T_{\mathfrak{P}_\Gamma}(X_1; E_\rho)$ and $\log T_{X_0}(X; E_\rho)$ are compared in the following proposition.

Proposition 2.1. *One has*

$$\log T_{\mathfrak{P}_\Gamma}(X; E_\rho) = \log T_{X_0}(X; E_\rho) + \sum_{P_j \in \mathfrak{P}_\Gamma} \log(t_{P_j}) \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) \lambda_{\rho,k}}{2} \right).$$

Proof. This follows by an application of a theorem of Kostant [Kos61] on nilpotent Lie algebra cohomology. For $p \in \{0, \dots, d\}$ define a representation of K on $\Lambda^p \mathfrak{p}^* \otimes V(\rho)$ by $\nu_p(\rho) := \Lambda^p \text{Ad}^* \otimes \rho$. Let $\tilde{E}_{\nu_p(\rho)} := G \times_{\nu_p(\rho)} K$, which is a homogeneous vector bundle over $\mathbb{H}^d = G/K$. Let Ω be the Casimir element of \mathfrak{g} . Then $-\Omega + \rho(\Omega)$ induces canonically a Laplace-type operator $\tilde{\Delta}_p(\rho)$ which acts on the smooth sections of $\tilde{E}_{\nu_p(\rho)}$. The heat semigroup of $e^{-t\tilde{\Delta}_p(\rho)}$ is canonically represented by a smooth function $H_t^{p,\rho} : G \rightarrow \text{End}(\Lambda^p \mathfrak{p}^* \otimes V_\rho)$, [MP12, section 4, section 7]. Let $h_t^{p,\rho} := \text{Tr } H_t^{p,\rho}$ and put

$$k_t^{p,\rho} := \sum_p (-1)^p p h_t^{p,\rho}.$$

Then by the definition of the regularized traces, one has

$$(2.8) \quad \int_{X_{\mathfrak{P}_\Gamma}(Y)} \left(\sum_{\gamma \in \Gamma} k_t^{p,\rho}(x^{-1} \gamma x) \right) dx = \alpha_{-1}(t; \rho) \log Y + K_{X;\mathfrak{P}_\Gamma}(t, \rho) + o(1),$$

as $Y \rightarrow \infty$, resp.

$$(2.9) \quad \int_{\pi^{-1}(X_{\mathfrak{P}_{\Gamma_0}}(Y))} \left(\sum_{\gamma \in \Gamma} k_t^{p,\rho}(x^{-1} \gamma x) \right) dx = \tilde{\alpha}_{-1}(t; \rho) \log Y + K_{X;X_0}(t, \rho) + o(1),$$

as $Y \rightarrow \infty$, where we use the notation (2.4). On the other hand, for $k = 0, \dots, n$ let $h_t^{\sigma_{\rho,k}} \in C^\infty(G)$ be defined as in [MP12, (8.7)]. If we apply the same considerations as in [MP14a, section 6] to the functions $h_t^{\sigma_{\rho,k}}$, then combining [MP12, Proposition 8.2], (2.8) and (2.9) we obtain

$$K_{X;X_0}(t, \rho) = K_{X;\mathfrak{P}_\Gamma}(t, \rho) - \sum_{P_j \in \mathfrak{P}_\Gamma} \log t_{P_j} \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) e^{-t\lambda_{\rho,k}^2}}{\sqrt{4\pi t}} \right).$$

Taking the Mellin transform, the Proposition follows. \square

Next, as in [Pfa13], for each $P_j \in \mathfrak{P}_\Gamma$ and for $Y > 0$ one can define the regularized analytic torsion $T(F_{P_j;\Gamma}(Y), \partial F_{P_j;\Gamma}(Y); E_\rho)$ of $F_{P_j;\Gamma}(Y)$ and the bundle $E_\rho|_{F_{P_j;\Gamma}(Y)}$, where one takes relative boundary conditions. For different Y , these torsions are compared by the following gluing formula.

Lemma 2.2. *Let $c(n) \in \mathbb{R}$ be as in [Pfa13, equation 15.10]. Then for $Y_1 > 0$ and $Y_2 > 0$ one has*

$$\begin{aligned} & \log T(F_{P_j;\Gamma}(Y_1), \partial F_{P_j;\Gamma}(Y_1); E_\rho) - c(n) \operatorname{vol}(\partial F_{P_j;\Gamma}(Y_1)) \operatorname{rk}(E_\rho) \\ &= \log T(F_{P_j;\Gamma}(Y_2), \partial F_{P_j;\Gamma}(Y_2); E_\rho) - c(n) \operatorname{vol}(\partial F_{P_j;\Gamma}(Y_2)) \operatorname{rk}(E_\rho) \\ &+ \sum_{k=0}^n (-1)^{k+1} \lambda_{\rho,k} \dim(\sigma_{\rho,k}) (\log(Y_2) - \log(Y_1)). \end{aligned}$$

Proof. This follows immediately from [Pfa13, Corollary 15.4, equation (15.11) and Corollary 16.2]. \square

We let \overline{X} denote the Borel-Serre compactification of X and we let $\tau_{Eis}(X; E_\rho)$ be the Reidemeister torsion of \overline{X} with coefficients in E_ρ , defined as in [Pfa13, section 9]. For simplicity, we assume that Γ is normal in Γ_0 . Then for the torsion $T_{X_0}(X; E_\rho)$, the main result of [Pfa13] can be restated as follows.

Proposition 2.3. *For the analytic torsion $T_{X_0}(X; E_\rho)$ one has*

$$\begin{aligned} \log \tau_{Eis}(X; E_\rho) &= \log T_{X_0}(X; E_\rho) - \frac{1}{4} \sum_{k=0}^{d-1} (-1)^k \log |\lambda_{\rho,k}| \dim H^k(\partial \overline{X}; E_\rho) \\ &- \sum_{l=1}^{\kappa(\Gamma_0)} \# \{P_j \in \mathfrak{P}_\Gamma : \gamma_j P_j \gamma_j^{-1} = P_{0,l}\} \left(\log T(F_{P_{0,l};\Gamma}(1), \partial F_{P_{0,l};\Gamma}(1); E_\rho) \right. \\ &\left. - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma}(1)) \operatorname{rk}(E_\rho) \right). \end{aligned}$$

Proof. By [Pfa13, Theorem 1.1] and by Proposition 2.1 we have

$$\begin{aligned} \log \tau_{Eis}(X; E_\rho) &= \log T_{X_0}(X; E_\rho) + \sum_{P_j \in \mathfrak{P}_\Gamma} \log(t_{P_j}) \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) \lambda_{\rho,k}}{2} \right) \\ &- \sum_{P_j \in \mathfrak{P}_\Gamma} (\log T(F_{P_j;\Gamma}(1), \partial F_{P_j;\Gamma}(1); E_\rho) - c(n) \operatorname{vol}(\partial F_{P_j;\Gamma}(1)) \operatorname{rk}(E_\rho)) \\ &- \frac{1}{4} \sum_{k=0}^{d-1} (-1)^k \log |\lambda_{\rho,k}| \dim H^k(\partial \overline{X}; E_\rho) \end{aligned}$$

where we recall that the regularized analytic torsion used in [Pfa13, Theorem 1.1] is the torsion denoted $T_{\mathfrak{P}_\Gamma}(X; E_\rho)$ here. Using that Γ is normal in Γ_0 , it easily follows from the definition of t_{P_j} that for each $P_j \in \mathfrak{P}_\Gamma$ one has a canonical isometry $\iota_{P_j; P_{0,l(j)}} : F_{P_j;\Gamma}(1) \cong$

$F_{P_{0,l(j)};\Gamma}(t_{P_j})$. It is easy to see that also $\iota_{P_j;P_{0,l(j)}}^* \left(E_\rho|_{F_{P_{0,l(j)};\Gamma}(t_{P_j})} \right)$ is isometric to $E_\rho|_{F_{P_j;\Gamma}(1)}$. Thus we have

$$\begin{aligned} & \log T(F_{P_j;\Gamma}(1), \partial F_{P_j;\Gamma}(1); E_\rho) - c(n) \operatorname{vol}(\partial F_{P_j;\Gamma}(1)) \operatorname{rk}(E_\rho) \\ &= \log T(F_{P_{0,l(j)};\Gamma}(t_{P_j}), \partial F_{P_{0,l(j)};\Gamma}(t_{P_j}); E_\rho) - c(n) \operatorname{vol}(\partial F_{P_{0,l(j)};\Gamma}(t_{P_j})) \operatorname{rk}(E_\rho). \end{aligned}$$

Applying Lemma 2.2, the Proposition follows. \square

Although the main topic of this paper is the behaviour of cohomological torsion of congruence subgroups of Bianchi groups under a variation of the local system, we now state the following limit multiplicity formula for twisted Reidemeister torsion in arithmetic hyperbolic congruence towers of arbitrary odd dimension, since the latter is an easy corollary.

Corollary 2.4. *Let $G := \operatorname{SO}^0(d, 1)$, d odd, and for $q \in \mathbb{N}$ let $\Gamma(q) := \ker(G(\mathbb{Z}) \rightarrow G(\mathbb{Z}/q\mathbb{Z}))$ denote the principal congruence subgroup of level q . Let $X_q := \Gamma(q) \backslash \mathbb{H}^d$. Then for any $\rho \in \operatorname{Rep}(G)$ with $\rho \neq \rho_\theta$ one has*

$$\lim_{q \rightarrow \infty} \frac{\log \tau_{\text{Eis}}(X_q; E_\rho)}{\operatorname{vol}(X_q)} = t_{\mathbb{H}^d}^{(2)}(\rho),$$

where $t_{\mathbb{H}^d}^{(2)}(\rho)$ is the L^2 -invariant associated to ρ and \mathbb{H}^d which is defined as in [BV13], [MP14a] and which is never zero. The same holds for every sequence $X_{\mathfrak{a}}$ of arithmetic hyperbolic 3-manifolds associated to principal congruence subgroups $\Gamma(\mathfrak{a})$ of Bianchi groups if $N(\mathfrak{a}) \rightarrow \infty$.

Proof. First we assume that $G = \operatorname{SO}^0(d, 1)$. Let $\Gamma_0 := G(\mathbb{Z})$ and $X_0 := \Gamma_0 \backslash \mathbb{H}^d$. By [MP14a, Corollary 1.3], for the analytic torsion $T_{X_0}(X_q; E_\rho)$ one has

$$(2.10) \quad \lim_{q \rightarrow \infty} \frac{\log T_{X_0}(X_q; E_\rho)}{\operatorname{vol}(X_q)} = t_{\mathbb{H}^d}^{(2)}(\rho),$$

as $q \rightarrow \infty$. Next it is well-known that for the number $\kappa(\Gamma(q))$ of cusps of $\Gamma(q)$ one has

$$(2.11) \quad \lim_{q \rightarrow \infty} \frac{\kappa(\Gamma(q))}{\operatorname{vol}(X_q)} = 0,$$

see for example [MP14a, Proposition 8.6]. For $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ we let $\Lambda_{\Gamma(q)}(P_{0,l}) := \log((\Gamma(q) \cap N_{P_{0,l}}))$, which is a lattice in $\mathfrak{n}_{P_{0,l}}$. By a result of Deitmar and Hoffmann [DH99, Lemma 4], for each $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$, there exists a finite set of lattices $\mathcal{L}_{P_{0,l}} = \{\Lambda_1(P_{0,l}), \dots, \Lambda_m(P_{0,l})\}$ in $\mathfrak{n}_{P_{0,l}}$ such that for each $q \in \mathbb{N}$ the lattice $\Lambda_{\Gamma(q)}(P_{0,l})$ arises by scaling one of the lattices $\Lambda_j(P_{0,l})$, $j = 1, \dots, m$, see [MP14a, Lemma 10.1]. For $\Lambda_j(P_{0,l}) \in \mathcal{L}_{P_{0,l}}$ we let $T_{\Lambda_j(P_{0,l})} := \Lambda_j(P_{0,l}) \backslash \mathfrak{n}_{P_{0,l}}$, equipped with the flat metric (2.1) restricted to $\mathfrak{n}_{P_{0,l}}$ which we shall denote by $g_{\Lambda_j(P_{0,l})}$. Then we let $F_{\Lambda_j(P_{0,l})}(1) := [1, \infty) \times \Lambda_j(P_{0,l})$, equipped with the metric $y^{-2}(dy^2 + g_{\Lambda_j(P_{0,l})})$, $y \in [1, \infty)$. If $\Lambda_{\Gamma(q)}(P_{0,l}) = \mu_{q;P_{0,l}} \Lambda_j(P_{0,l})$ with $\Lambda_j(P_{0,l}) \in \mathcal{L}_{P_{0,l}}$ and $\mu_{q;P_{0,l}} \in (0, \infty)$, then by

Lemma 2.2 one has

$$\begin{aligned} & \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \operatorname{rk}(E_\rho) \\ &= \log T \left(F_{\Lambda_j(P_{0,l})}(1), \partial F_{\Lambda_j(P_{0,l})}(1); E_\rho \right) - c(n) \operatorname{vol}(\Lambda_j(P_{0,l})) \operatorname{rk}(E_\rho) \\ & \quad + \log \mu_{q;P_{0,l}} \sum_{k=0}^n (-1)^k \lambda_{\rho,k} \dim(\sigma_{\rho,k}). \end{aligned}$$

We can obviously estimate $\mu_{q;P_{0,l}} \leq C_1 [\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]$, where C_1 is a constant which is independent of q . Thus there exists a constant C_2 such that for all q one can estimate

$$\begin{aligned} & \left| \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \operatorname{rk}(E_\rho) \right| \\ & \leq C_2 \log [\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]. \end{aligned}$$

For each $l \in \{1, \dots, \kappa(\Gamma_0)\}$ one has

$$\frac{\#\{P_j \in \mathfrak{P}_{\Gamma(q)} : \gamma_j P_j \gamma_j^{-1} = P_{0,l}\}}{[\Gamma_0 : \Gamma(q)]} = \frac{\#(\Gamma(q) \backslash \Gamma_0 / (\Gamma_0 \cap P_{0,l}))}{[\Gamma_0 : \Gamma(q)]} \leq \frac{1}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}.$$

Thus for all q one can estimate

$$\begin{aligned} & \sum_{l=1}^{\kappa(\Gamma_0)} \frac{\#\{P_j \in \mathfrak{P}_{\Gamma(q)} : \gamma_j P_j \gamma_j^{-1} = P_{0,l}\}}{[\Gamma_0 : \Gamma(q)]} \\ & \quad \times \left| \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \operatorname{rk}(E_\rho) \right| \\ (2.12) \quad & \leq C_2 \sum_{l=1}^{\kappa(\Gamma_0)} \frac{\log [\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}. \end{aligned}$$

Since $\cap_q \Gamma(q) = \{1\}$, $[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]$ as $q \rightarrow \infty$ and thus the last term in (2.12) goes to zero as q tends to infinity. Since $\dim H^k(\partial \overline{X}_q) = O(\kappa(\Gamma_q))$, the corollary follows by applying Proposition 2.3, equation (2.10) and equation (2.11). For a sequence X_a associated to principal congruence subgroups of Bianchi groups, one argues in the same way. \square

3. ANALYTIC AND COMBINATORIAL TORSION FOR CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

From now on, we let $G = \operatorname{Spin}(3, 1) = \operatorname{SL}_2(\mathbb{C})$, $K = \operatorname{Spin}(3) = \operatorname{SU}(2)$. We take P to be the standard parabolic subgroup of G consisting of all upper triangular matrices in G . Then the unipotent radical N_P of P is given by all upper triangular matrices whose diagonal entries are one. Moreover, we let M_P and A_P denote the subgroups of $\operatorname{SL}_2(\mathbb{C})$ defined by

$$(3.1) \quad M_P := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi] \right\}; \quad A_P := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in (0, \infty) \right\}.$$

Then $P = M_P A_P N_P$. Let \mathfrak{n}_P denote the Lie algebra of N_P and let \mathfrak{a}_P denote the Lie algebra of A_P . For $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ we let $\sigma_k : M_P \rightarrow \mathbb{C}$, $\xi_\lambda : A_P \rightarrow \mathbb{C}$ be defined by

$$\sigma_k \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) := e^{ik\theta}; \quad \xi_\lambda \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) := t^{2\lambda}.$$

Then the assignment $\lambda \rightarrow \xi_\lambda$ is consistent with our earlier identification $\mathbb{C} \cong (\mathfrak{a}_P^*)_{\mathbb{C}}$. The group K_∞ acts on $\mathfrak{g}/\mathfrak{k}$ by Ad and we let K_∞ act on $\mathfrak{a}_P \oplus \mathfrak{n}_P$ by using the canonical identification $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{a}_P \oplus \mathfrak{n}_P$.

Let $F := \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic number field and let d_F be its class number. Let \mathcal{O}_D denote the ring of integers of F . Let \mathcal{O}_D^* be the group of units of \mathcal{O}_D , i.e. $\mathcal{O}_D^* = \{\pm 1\}$ for $D \neq 1, 3$, $\mathcal{O}_D^* = \{\pm 1, \pm \sqrt{-1}\}$ for $D = 1$, $\mathcal{O}_D^* = \{\pm 1, \pm \frac{1 \pm \sqrt{-3}}{2}\}$ for $D = 3$. Let $\Gamma_D := \mathrm{SL}_2(\mathcal{O}_D)$. The quotient $X_0 := \Gamma_D \backslash \mathbb{H}^3$ is a hyperbolic orbifold of finite volume, see for example [EGM98]. We have

$$(3.2) \quad \Gamma_D \cap N_P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathcal{O}_D \right\}$$

We fix a set $\mathfrak{P}_{\Gamma_D} = \{P_{0,1}, \dots, P_{0,\kappa(\Gamma_D)}\}$ of representatives of Γ_D -cuspidal parabolic subgroups of G , where we require each parabolic to be F -rational and where we assume that $P_{0,1} = P$. Let $\mathbb{P}^1(F)$ be the one-dimensional projective space of F . As usual, we write ∞ for the element $[1, 0] \in \mathbb{P}^1(F)$. Then $\mathrm{SL}_2(F)$ acts transitively on $\mathbb{P}^1(F)$ and P is the stabilizer of ∞ . In particular for each subgroup Γ of Γ_D one has $\kappa(\Gamma) = \#(\Gamma \backslash G/P) = \#(\Gamma \backslash \mathbb{P}^1(F))$. Moreover, by [EGM98, Chapter 7.2, Theorem 2.4], one has $\kappa(\Gamma_D) = d_F$. Let $\{\eta_l : l = 1, \dots, \eta_{d_F}\}$ denote fixed representatives of $\Gamma_D \backslash \mathbb{P}^1(F)$ such that $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ is the stabilizer of η_l in $\mathrm{SL}_2(\mathbb{C})$ for each l .

For \mathfrak{a} a non-zero ideal of \mathcal{O}_D we let $\Gamma(\mathfrak{a})$ denote the principal congruence subgroup of Γ_D of level \mathfrak{a} . This group is normal in Γ_D ; moreover, for $N(\mathfrak{a})$ sufficiently large ($N(\mathfrak{a}) \geq 3$ in the case $\mathcal{O}_D^* = \pm 1$), the group $\Gamma(\mathfrak{a})$ is neat. We shall assume from now on that this is the case. Then $X_{\mathfrak{a}} := \Gamma(\mathfrak{a}) \backslash \mathbb{H}^3$ is a hyperbolic 3-manifold of finite volume. According to the previous section, for $\rho \in \mathrm{Rep}(G)$ with $\rho \neq \rho_\theta$, we can define the analytic torsion $\log T_{X_0}(X_{\mathfrak{a}}; E_\rho)$ of $X_{\mathfrak{a}}$ with coefficients in E_ρ with respect to the choice of truncation parameters coming from the covering $\pi : X_{\mathfrak{a}} \rightarrow X_0$ and the choice of \mathfrak{P}_{Γ_D} .

We shall now simplify the formula in (2.3) a bit further for the specific manifolds $X_{\mathfrak{a}}$. Let \mathfrak{b} be a non-zero ideal of \mathcal{O}_D . Taking the identification (3.2), we shall regard \mathfrak{b} as a lattice in \mathfrak{n}_P . We denote this lattice by $\Lambda_P(\mathfrak{b})$ and we let $T_{\Lambda_P(\mathfrak{b})} := \exp(\Lambda_P(\mathfrak{b})) \backslash N_P$ denote the corresponding torus. As above, for $r > 0$ we let $F_{\Lambda_P(\mathfrak{b})}(r) := [r, \infty) \times T_{\Lambda_P(\mathfrak{b})}$ denote the corresponding cusp. We fix ideals \mathfrak{s}_l , $l = 1, \dots, d_F$ in \mathcal{O}_D which represent the class group of F . Then we have:

Proposition 3.1. *There exist $n_{1,\Gamma(\mathfrak{a})}, \dots, n_{d_F,\Gamma(\mathfrak{a})} \in \mathbb{N} \cup \{0\}$, with $n_{1,\Gamma(\mathfrak{a})} + \dots + n_{d_F,\Gamma(\mathfrak{a})} = d_F$, and there exist $\mu_{1,\Gamma(\mathfrak{a})}, \dots, \mu_{d_F,\Gamma(\mathfrak{a})} \in (0, \infty)$ such that*

$$\begin{aligned} \log \tau_{Eis}(X_{\mathfrak{a}}; E_{\rho}) &= \log T_{X_0}(X_{\mathfrak{a}}; E_{\rho}) - \sum_{l=1}^{d_F} \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})} \left(n_{l,\Gamma(\mathfrak{a})} \log T(F_{\Lambda_P(\mathfrak{s}_l)}(1), \partial F_{\Lambda_P(\mathfrak{s}_l)}(1); E_{\rho}) \right. \\ &\quad \left. - n_{l,\Gamma(\mathfrak{a})} c(1) \operatorname{vol}(\Lambda_P(\mathfrak{s}_l)) \operatorname{rk}(E_{\rho}) + (\lambda_{\rho,1} - \lambda_{\rho,0}) \mu_{l,\Gamma(\mathfrak{a})} \log \mu_{l,\Gamma(\mathfrak{a})} \right) \\ &\quad - \frac{\kappa(\Gamma(\mathfrak{a}))}{2} (\lambda_{\rho,0} - \lambda_{\rho,1}). \end{aligned}$$

Here the $n_{1,\Gamma(\mathfrak{a})}, \dots, n_{d_F,\Gamma(\mathfrak{a})} \in \mathbb{N} \cup \{0\}$ and the $\mu_{1,\Gamma(\mathfrak{a})}, \dots, \mu_{d_F,\Gamma(\mathfrak{a})} \in (0, \infty)$ depend on $\Gamma(\mathfrak{a})$, but not on the representation ρ .

Proof. Let $\{\eta_l : l = 1, \dots, \eta_{d_F}\}$ denote fixed representatives of $\Gamma_D \backslash \mathbb{P}^1(F)$ such that $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ is the stabilizer of η_l in $\operatorname{SL}_2(\mathbb{C})$ for each l . For each $l = 1, \dots, d_F$ we fix $B_{\eta_l} \in \operatorname{SL}_2(F)$ with $B_{\eta_l} \eta_l = \infty$. Then $P_{0,l} =: P_{\eta_l} = B_{\eta_l}^{-1} P B_{\eta_l}$. Let $(\Gamma_D)_{\eta_l} = \Gamma_D \cap P_{\eta_l}$ resp. $\Gamma(\mathfrak{a})_{\eta_l} = \Gamma(\mathfrak{a}) \cap P_{\eta_l}$ be the stabilizers of η_l in Γ_D resp. $\Gamma(\mathfrak{a})$. One has

$$(3.3) \quad B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} = \left\{ J(B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} \cap N) : J \in \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha \in \mathcal{O}_D^* \right\} \right\}.$$

In particular, one has $B_{\eta_l} \Gamma(\mathfrak{a})_{\eta_l} B_{\eta_l}^{-1} \cap P = B_{\eta_l} \Gamma(\mathfrak{a})_{\eta_l} B_{\eta_l}^{-1} \cap N$ for $N(\mathfrak{a})$ sufficiently large. Write $B_{\eta_l} = \begin{pmatrix} \alpha_l & \beta_l \\ \gamma_l & \delta_l \end{pmatrix} \in \operatorname{SL}_2(F)$ and let \mathfrak{u}_l be the \mathcal{O}_D -module generated by γ_l and δ_l . Then one has

$$B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} \cap N = \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : \omega \in \mathfrak{u}_l^{-2} \right\}; \quad B_{\eta_l} \Gamma(\mathfrak{a})_{\eta_l} B_{\eta_l}^{-1} \cap N = \left\{ \begin{pmatrix} 1 & \omega' \\ 0 & 1 \end{pmatrix} : \omega' \in \mathfrak{a} \mathfrak{u}_l^{-2} \right\},$$

where the first equality is proved in [EGM98, Chapter 8.2, Lemma 2.2] and where the second equality can be proved using the same arguments. Thus one has

$$[B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} \cap N : B_{\eta_l} \Gamma(\mathfrak{a})_{\eta_l} B_{\eta_l}^{-1} \cap N] = N(\mathfrak{a}).$$

Thus by (3.3), for each $l = 1, \dots, d_F$ and $N(\mathfrak{a})$ sufficiently large one has $[(\Gamma_D)_{\eta_l} : \Gamma(\mathfrak{a})_{\eta_l}] = \#(\mathcal{O}_D^*)N(\mathfrak{a})$ and so one has

$$\#\{P_j \in \mathfrak{P}_{\Gamma(\mathfrak{a})} : \gamma_j P_j \gamma_j^{-1} = P_{0,l}\} = \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})}$$

For each l there is a constant $\kappa_l > 0$ which depends only on B_{η_l} such that one has a canonical isometry $\iota : F_{P_{0,l};\Gamma(\mathfrak{a})}(1) \cong F_{\Lambda_P(\kappa_l \mathfrak{a} \mathfrak{u}_l^{-2})}(1)$ which induces an isomtery $\iota^* E_{\rho}|_{F_{\Lambda_P(\kappa_l \mathfrak{a} \mathfrak{u}_l^{-2})}(1)} \cong E_{\rho}|_{F_{P_{0,l};\Gamma(\mathfrak{a})}(1)}$. Next, there exists a unique map σ from $\{1, \dots, d_F\}$ into itself such that for each $l = 1, \dots, d_F$ there exists a $\tilde{\mu}_{l,\Gamma(\mathfrak{a})} \in F^*$ with $\mathfrak{a} \mathfrak{u}_l^{-2} = \tilde{\mu}_{l,\Gamma(\mathfrak{a})} \mathfrak{s}_{\sigma(l)}$. We let $\mu_{l,\Gamma(\mathfrak{a})} := \kappa_l |\tilde{\mu}_{l,\Gamma(\mathfrak{a})}|$. Then it follows that there is a canonical isometry $\iota : F_{P_{0,l};\Gamma(\mathfrak{a})}(1) \cong F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(\mu_{l,\Gamma(\mathfrak{a})}^{-1})$

which extends to an isometry $\iota^* E_\rho|_{F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(\mu_{l,\Gamma(\mathfrak{a})}^{-1})} \cong E_\rho|_{F_{P_{0,l};\Gamma(\mathfrak{a})}(1)}$. Thus it follows from Lemma 2.2 that

$$\begin{aligned} & \log T(F_{P_{0,l};\Gamma(\mathfrak{a})}(1), \partial F_{P_{0,l};\Gamma(\mathfrak{a})}(1); E_\rho) - c(1) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(\mathfrak{a})}(1)) \operatorname{rk}(E_\rho) \\ &= \log T\left(F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1), \partial F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1); E_\rho\right) - c(1) \operatorname{vol}\left(\partial F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1)\right) + \log \mu_{l,\Gamma(\mathfrak{a})}(\lambda_{\rho,1} - \lambda_{\rho,0}). \end{aligned}$$

For each $l = 1, \dots, d_F$ we let $n_{l,\Gamma(\mathfrak{a})} := \#\{\sigma^{-1}(l)\}$. Then, since $\dim(\sigma_{\rho,k}) = 1$ in the case of $G = \operatorname{SL}_2(\mathbb{C})$, the Proposition follows from Proposition 2.3. \square

We let ρ_1 denote the standard representation of $\operatorname{SL}_2(\mathbb{C})$ on \mathbb{C}^2 and for $m \in \mathbb{N}$ we consider the representation $\rho_m := \operatorname{Symm}^m \rho_1$ on $V(m) := \operatorname{Symm}^m \mathbb{C}^2$. We can now deduce the following result on the growth of the torsion $\tau_{Eis}(X_{\mathfrak{a}}; E(\rho_m))$ if $m \rightarrow \infty$.

Proposition 3.2. *Let \mathfrak{a} and \mathfrak{a}_0 be two ideals in \mathcal{O}_D such that $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{a}_0)$ are neat and such that, in the notation of the previous proposition, one has $n_{l,\Gamma(\mathfrak{a})} = n_{l,\Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. Then one has*

$$\begin{aligned} & \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{\#(\mathcal{O}_D^*)N(\mathfrak{a}_0)} \log \tau_{Eis}(X_{\mathfrak{a}}; E(\rho_m)) - \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})} \log \tau_{Eis}(X_{\mathfrak{a}_0}; E(\rho_m)) \\ &= - \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)][\Gamma_D : \Gamma(\mathfrak{a})]}{\#\mathcal{O}_D^* \pi} \left(\frac{1}{N(\mathfrak{a}_0)} - \frac{1}{N(\mathfrak{a})} \right) \operatorname{vol}(\Gamma_D \backslash \mathbb{H}^3) m^2 + O(m \log m), \end{aligned}$$

as $m \rightarrow \infty$.

Proof. In the notation of [MP12], the representation ρ_m is the representation of highest weight $(m/2, m/2)$. If Γ is a neat finite index subgroup of Γ_D and if we let $X := \Gamma \backslash \mathbb{H}^3$, then specializing [MP12, Theorem 1.1] to the present situation, we obtain

$$\lim_{m \rightarrow \infty} \log T_{X_0}(X; E_{\rho_m}) = -\frac{1}{\pi} \operatorname{vol}(X) m^2 + O(m \log m),$$

as $m \rightarrow \infty$. Moreover, one has $\lambda_{\rho_m,0} = (m+1)/2$ and $\lambda_{\rho_m,1} = m/2$. Thus the proposition follows from the previous Proposition 3.1. \square

There is a constant $C'_1(\Gamma)$ such that $\Gamma(\mathfrak{a})$ is neat for all ideals \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) \geq C'_1(\Gamma)$. If $\mathcal{O}_D^* = \{\pm 1\}$, one has $C'_1(\Gamma) = 3$. Next we remark that by the requirement $n_{1,\Gamma(\mathfrak{a})} + \dots + n_{d_F}(\mathfrak{a}) = d_F$ there is a finite set \mathcal{A} of ideals of \mathcal{O}_D such that $N(\mathfrak{a}_0) \geq C'_1(\Gamma)$ for each $\mathfrak{a}_0 \in \mathcal{A}$ and such that for each non-zero ideal \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) \geq C'_1(\Gamma)$ there is an $\mathfrak{a}_0 \in \mathcal{A}$ such that $n_{l,\Gamma(\mathfrak{a})} = n_{l,\Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. We let $\tilde{C}_1(\Gamma) := \max\{N(\mathfrak{a}_0) : \mathfrak{a}_0 \in \mathcal{A}\}$ and we let $C_1(\Gamma) := \max\{\tilde{C}_1(\Gamma), C'_1(\Gamma)\}$.

4. TORSION IN COHOMOLOGY AND REIDEMEISTER TORSION

We keep the notation of the preceding section. Let $\Lambda(m) := \operatorname{Symm}^m \mathcal{O}_D$. Then $\Lambda(m)$ is a lattice in $V(m)$ which is preserved by $\Gamma(D)$; we denote the representation of $\Gamma(D)$ on $\operatorname{Aut}_{\mathbb{Z}}(\Lambda(m))$ by $\rho_{m;\mathbb{Z}}$. Let $\check{\Lambda}(m) := \operatorname{Hom}_{\mathbb{Z}}(\Lambda(m), \mathbb{Z})$ denote the dual lattice of $\Lambda(m)$ and let $\check{\rho}_{m;\mathbb{Z}}$ denote the contragredient representations of $\Gamma(D)$ on $\check{\Lambda}_m$. We let $\bar{\rho}_{\mathbb{Z};m} := \rho_{\mathbb{Z};m} \oplus \check{\rho}_{\mathbb{Z};m}$ denote the corresponding integral representation of $\Gamma(D)$ on $\bar{\Lambda}(m) := \Lambda(m) \oplus \check{\Lambda}(m)$. We let

$\bar{\rho}_m$ denote the corresponding real representation of Γ on $\bar{V}(m) := V(m) \oplus V(m)^*$. Over \mathbb{R} , the representation ρ_m is self-dual, i.e. one has $\bar{\rho}_m \cong \rho_m \oplus \rho_m$. In particular, no irreducible summand of $\bar{\rho}_m$ is invariant under the Cartan involution θ . Let Γ be any neat, finite index subgroup of $\Gamma(D)$. Then we regard each $\bar{\Lambda}(m)$ as a Γ -module. Let $H^*(\Gamma, \bar{\Lambda}(m))$ be the cohomology groups of Γ with coefficients in $\bar{\Lambda}(m)$. These groups are finitely generated abelian groups and thus they admit a decomposition

$$H^*(\Gamma, \bar{\Lambda}(m)) = H^*(\Gamma, \bar{\Lambda}(m))_{free} \oplus H^*(\Gamma, \bar{\Lambda}(m))_{tors},$$

where $H^*(\Gamma, \bar{\Lambda}(m))_{free}$ are free, finite-rank \mathbb{Z} -modules and where $H^*(\Gamma, \bar{\Lambda}(m))_{tors}$ are finite abelian groups. Let $X := \Gamma \backslash \mathbb{H}^3$ and let \bar{X} denote the Borel-Serre compactification of X . The latter is homotopy equivalent to X . In particular, $\bar{\rho}_{\mathbb{Z}, m}$ defines an integral local system $\mathcal{L}(m)$ over \bar{X} . Since the universal covering of X resp. \bar{X} is contractible, one has a canonical isomorphism $H^*(\Gamma, \bar{\Lambda}(m)) \cong H^*(\bar{X}, \mathcal{L}(m))$. Let $E_{\bar{\rho}_m}$ denote the flat vector bundle over X resp. \bar{X} corresponding to $\bar{\rho}_m$ and let $H^*(\bar{X}; E_{\bar{\rho}_m})$ denote the singular cohomology groups of \bar{X} with coefficients in $E_{\bar{\rho}_m}$. Then $H^*(\Gamma, \bar{\Lambda}(m))_{free}$ is a lattice in $H^*(\bar{X}; E_{\bar{\rho}_m})$.

We now recall the description of the canonical bases in the cohomology $H^*(\bar{X}; E_{\bar{\rho}_m})$ which are used to define the Reidemeister torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$. For more details, we refer to [Pfa13]. We shall use the notation of [Pfa13, section 8]. Let $\partial\bar{X}$ denote the boundary of \bar{X} . If $\iota : \partial\bar{X} \rightarrow X$ denotes the inclusion map, the corresponding maps $\iota_k^* : H^k(\bar{X}; E_{\bar{\rho}_m}) \rightarrow H^k(\partial\bar{X}; E_{\bar{\rho}_m})$ in cohomology are injective for $k \in \{1, 2\}$, see [Pfa13, Lemma 8.3]. Thus the cohomology $H^*(\bar{X}; E_{\bar{\rho}_m})$ is completely described in terms of Eisenstein cohomology due to Harder [Har75]. For each $P_j \in \mathfrak{P}_\Gamma$ let $\mathcal{H}^k(\mathfrak{n}_{P_j}, \bar{V}(m))$ denote the harmonic forms of degree k in the Lie algebra cohomology complex of \mathfrak{n}_{P_j} with coefficients in $\bar{V}(m)$. We equip this space with the inner product induced by the restriction of the inner product (2.1) on \mathfrak{g} to \mathfrak{n}_{P_j} and the admissible inner product on $\bar{V}(m)$. Let $\sigma_{\bar{\rho}_m, 1}^- \in \hat{M}_P$ and $\lambda_{\bar{\rho}_m, 1}^- \in (-\infty, 0)$ resp. $\sigma_{\bar{\rho}_m, 2}^- \in \hat{M}_P$ and $\lambda_{\bar{\rho}_m, 2}^- \in (-\infty, 0)$ be defined as in [Pfa13, section 6]. In the present situation, we have $\sigma_{\bar{\rho}_m, 1}^- = \sigma_{-m-2}$ and $\lambda_{\bar{\rho}_m, 1}^- = -m/2$ resp. $\sigma_{\bar{\rho}_m, 2}^- = \sigma_{-m}$ and $\lambda_{\bar{\rho}_m, 2}^- = -(m+1)/2$. By the finite-dimensional Hodge theorem and a theorem of van Est one has a canonical isomorphism

$$(4.1) \quad H^k(\partial\bar{X}; E_{\bar{\rho}_m}) \cong \bigoplus_{P_j \in \mathfrak{P}_\Gamma} \mathcal{H}^k(\mathfrak{n}_{P_j}, \bar{V}(m)).$$

In degree 1, Kostant's theorem gives a splitting $\mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m)) = \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_- \oplus \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_+$. We let

$$H^1(\partial\bar{X}; E_{\bar{\rho}_m})_{\pm} := \bigoplus_{j=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_{\pm}.$$

Then out of the constant term matrix of the Eisenstein series one obtains a map

$$\underline{\mathbf{C}}(\sigma_{-m-2}, m/2) : H^1(\partial\bar{X}; E_{\bar{\rho}_m})_- \rightarrow H^1(\partial\bar{X}; E_{\bar{\rho}_m})_+.$$

We have $\dim \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m)) = 2 \dim \mathcal{H}^2(\mathfrak{n}_{P_j}; V(m)) = 2$ as well as $\dim \mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{\pm} = 2 \dim \mathcal{H}^1(\mathfrak{n}_{P_j}; V(m))_{\pm} = 2$. For each $P_j \in \mathfrak{P}_{\Gamma}$ let $\Phi_{i,j}^1$ be an orthonormal basis of $\mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}$ and let $\Phi_{i,j}^2$ be an orthonormal basis of $\mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))$. Then the set

$$\mathcal{B}^1(\Gamma; \bar{\rho}_m) := \{E(\Phi_{i,j}^1, m/2) : j = 1, \dots, \kappa(\Gamma) : i = 1, \dots, \dim \mathcal{H}^n(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}\}$$

forms a basis of $H^1(\bar{X}; E_{\bar{\rho}_m})$, where $E(\Phi_{i,j}^n, m/2)$ denotes again the Eisenstein series evaluated at $m/2$ as in [Pfa13, (7.3)] which is regular at this point by [Pfa13, Proposition 8.4]. Moreover, in degree 2 the set

$$\mathcal{B}^2(\Gamma; \bar{\rho}_m) := \{E(\Phi_{i,j}^2, (m+1)/2) : j = 1, \dots, \kappa(\Gamma) : i = 1, \dots, \dim \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))\}$$

forms a basis of $H^2(\bar{X}; E_{\bar{\rho}_m})$. Here $E(\Phi_{i,j}^2, (m+1)/2)$ denotes the Eisenstein series associated to $\Phi_{i,j}^2$ evaluated at $(m+1)/2$ which is again regular at this point. For $\Phi_{i,j}^1 \in \mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}$ one has

$$(4.2) \quad \iota_1^* E(\Phi_{i,j}^1, m/2) = \Phi_{i,j}^1 + \underline{\mathbf{C}}(\sigma_{-m-2}, m/2) \Phi_{i,j}^1$$

and for $\Phi_{i,j}^2 \in \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))$ one has

$$(4.3) \quad \iota_2^* E(\Phi_{i,j}^2, (m+1)/2) = \Phi_{i,j}^2.$$

By the definition of [Pfa13, section 9], the Reidemeister torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$ is taken using the above bases in the cohomology. In particular, following Bergeron and Venkatesh [BV13], the size of the groups $H^*(\Gamma, \bar{\Lambda}(m))_{tors}$ is related to the combinatorial torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$ in the following way. For $k \in \{1, 2\}$ we let $\text{vol}_{\mathcal{B}_k(\Gamma; \bar{\rho}_m)}(H^k(\Gamma; \bar{\Lambda}(m))_{free})$ denote the covolume of the lattice $H^k(\Gamma; \bar{\Lambda}(m))_{free}$ in $H^k(\bar{X}; E(\bar{\rho}_m))$ with respect to the inner product which arises by taking the basis $\mathcal{B}_k(\Gamma; \bar{\rho}_m)$ as an orthonormal basis. Then by [BV13, section 2.2] one has

$$(4.4) \quad \begin{aligned} \log \tau_{Eis}(X; E_{\bar{\rho}_m}) &= \log |H^1(\Gamma; \bar{\Lambda}(m))_{tors}| - \log |H^2(\Gamma; \bar{\Lambda}(m))_{tors}| \\ &\quad - \log \text{vol}_{\mathcal{B}^1(\Gamma; \bar{\rho}_m)}(H^1(X, \bar{\Lambda}(m))_{free}) + \log \text{vol}_{\mathcal{B}^2(\Gamma; \bar{\rho}_m)}(H^2(X, \bar{\Lambda}(m))_{free}) \end{aligned}$$

In the notation of [BV13], the term in the second line of (4.4) is called the regulator. We need to study the regulator further. We denote by $H^*(\partial \bar{X}; \bar{\Lambda}(m))$ resp. $H^*(\partial \bar{X}; E(\bar{\rho}_m))$ the cohomology of $\partial \bar{X}$ with coefficients in $\mathcal{L}(m)$ resp. $E(\bar{\rho}_m)$ restricted to $\partial \bar{X}$. Again there is a decomposition $H^*(\partial \bar{X}; \bar{\Lambda}(m)) = H^*(\partial \bar{X}; \bar{\Lambda}(m))_{free} \oplus H^*(\partial \bar{X}; \bar{\Lambda}(m))_{tors}$ and $H^*(\partial \bar{X}; \bar{\Lambda}(m))_{free}$ is a lattice in $H^*(\partial \bar{X}; E_{\bar{\rho}_m})$. It is easy to see that $H^1(\partial \bar{X}; \bar{\Lambda}(m))_{\pm} := H^1(\partial \bar{X}; \bar{\Lambda}(m))_{free} \cap H^1(\partial \bar{X}; E_{\bar{\rho}_m})_{\pm}$ are \mathbb{Z} -lattices in $H^1(\partial \bar{X}; E_{\bar{\rho}_m})_{\pm}$. Thus $H^1(\partial \bar{X}; \bar{\Lambda}(m))_{-} \oplus H^1(\partial \bar{X}; \bar{\Lambda}(m))_{+}$ is a \mathbb{Z} -sublattice of finite index in $H^1(\partial \bar{X}; \bar{\Lambda}(m))_{free}$. Moreover, with respect to the lattices $H^1(\partial \bar{X}; \bar{\Lambda}(m))_{\pm}$, the matrix $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ is \mathbb{Q} -rational. In the present case where there is no interior cohomology, this just follows from (4.2) and the fact that the map ι_1^* is defined over \mathbb{Q} . In other words, there exists $N \in \mathbb{N}$ such that $N \cdot \underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ defines a map

$$(4.5) \quad N \cdot \underline{\mathbf{C}}(\sigma_{-m-2}, m/2) : H^1(\partial \bar{X}; \bar{\Lambda}(m))_{-} \rightarrow H^1(\partial \bar{X}; \bar{\Lambda}(m))_{+}.$$

We let $d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)$ denote the smallest $N \in \mathbb{N}$ such that (4.5) holds. We point out that we do not treat the denominator of an Eisenstein cohomology class but only the denominator of the constant term matrix. However, this is sufficient for our purposes due to the following Lemma.

Lemma 4.1. *One can estimate*

$$\begin{aligned} \log \text{vol}_{\mathcal{B}^1(\Gamma; \bar{\rho}_m)}(H^1(X, \bar{\Lambda}(m)))_{free} &\leq \log |H^2(X, \bar{\Lambda}(m))|_{tors} + \kappa(\Gamma) \log (d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)) \\ &+ \kappa(\Gamma) \log ([H^1(\partial \bar{X}; \bar{\Lambda}(m))_{free} : H^1(\partial \bar{X}; \bar{\Lambda}(m))_- \oplus H^1(\partial \bar{X}; \bar{\Lambda}(m))_+]). \end{aligned}$$

Proof. Let $k := [H^1(\partial \bar{X}; \bar{\Lambda}(m))_{free} : H^1(\partial \bar{X}; \bar{\Lambda}(m))_- \oplus H^1(\partial \bar{X}; \bar{\Lambda}(m))_+]$. We define a free \mathbb{Z} -submodule $A(\bar{\rho}_m)$ of $H^1(\partial \bar{X}; \bar{\Lambda}(m))_- \oplus H^1(\partial \bar{X}; \bar{\Lambda}(m))_+$ by

$$\begin{aligned} A(\bar{\rho}_m) \\ := \{ \eta_i + \underline{\mathbb{C}}(\sigma_{-m-2}, m/2) \eta_i : \eta_i \in H^1(\partial \bar{X}; \bar{\Lambda}(m))_- : \underline{\mathbb{C}}(\sigma_{-m-2}, m/2) \eta_i \in H^1(\partial \bar{X}; \bar{\Lambda}(m))_+ \}. \end{aligned}$$

Let $\text{pr}_{free} : H^1(\partial \bar{X}; \bar{\Lambda}(m)) \rightarrow H^1(\partial \bar{X}; \bar{\Lambda}(m))_{free}$ be the projection. Then by (4.2) the lattice $k \cdot \text{pr}_{free} i_1^* H^1(\bar{X}; \bar{\Lambda}(m))_{free}$ is a \mathbb{Z} -sublattice of $A(\bar{\rho}_m)$ whose \mathbb{Z} -rank is equal to that of $A(\bar{\rho}_m)$. Thus the quotient $A(\bar{\rho}_m) / (k \cdot \text{pr}_{free} i_1^* H^1(\bar{X}; \bar{\Lambda}(m))_{free})$ embeds into $H^2(\bar{X}, \partial \bar{X}; \bar{\Lambda}(m))_{tors}$ by the long exact cohomology sequence. By Poincaré duality, [Wal66, page 223-224] and the universal coefficient theorem one has

$$H^2(\bar{X}, \partial \bar{X}; \bar{\Lambda}(m))_{tors} \cong H_1(\bar{X}; \bar{\Lambda}(m))_{tors} \cong H^2(\bar{X}; \bar{\Lambda}(m))_{tors},$$

where in the last isomorphism we used that $\bar{\Lambda}(m)$ was self-dual over \mathbb{Z} . On the other hand, if $\pi_- : A(\bar{\rho}_m) \rightarrow H^1(\partial \bar{X}; \bar{\Lambda}(m))_-$ is the projection, then by the definition of $d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)$ and the fact that $\text{rk}_{\mathbb{Z}} H^1(\partial \bar{X}; \bar{\Lambda}(m))_- = \kappa(\Gamma)$, the order of the quotient $H^1(\partial \bar{X}; \bar{\Lambda}(m))_- / \pi_- A(\bar{\rho}_m)$ can be estimated as

$$|H^1(\partial \bar{X}; \bar{\Lambda}(m))_- / \pi_- A(\bar{\rho}_m)| \leq d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)^{\kappa(\Gamma)}.$$

Thus the first estimate follows easily from the definition of $\text{vol}_{\mathcal{B}^1(\Gamma; \bar{\rho}_m)}(H^1(X, \bar{\Lambda}(m)))_{free}$. \square

Lemma 4.2. *We have :*

$$\log \text{vol}_{\mathcal{B}^2(\Gamma; \bar{\rho}_m)}(H^2(X, \bar{\Lambda}(m)))_{free} \leq \log |H^1(X, \bar{\Lambda}(m))|_{tors}.$$

Proof. Similar to that of the previous lemma. \square

5. THE ADELIC INTERTWINING OPERATORS

In this section and the next one, we want to establish an estimate of $d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)$ by working adelically. We let $G := \text{SL}_2$ regarded as an algebraic group over F . Also, for notational convenience we shall write $K_\infty := \text{SU}(2)$. Let \mathbb{A} denote the adèle ring of F and let \mathbb{A}_f be the finite adeles. For a linear algebraic group H defined over F let $H(\mathbb{A})$ denote its adelic points. For v a finite place we let F_v denote the completion of F at v , we let \mathcal{O}_v denote the integers in F_v and we let $\pi_v \in \mathcal{O}_v$ be a fixed uniformizer. We let Γ be a fixed

neat congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_D)$ of level $\mathfrak{a} = \prod_{v \text{ finite}} \mathfrak{p}_v^{n_v}$, where \mathfrak{p}_v is the prime ideal corresponding to v .

Let P be the parabolic subgroup of G consisting of upper triangular matrices and let T denote the set of diagonal matrices of determinant one. Let N_P denote the upper triangular matrices with 1 as diagonal entries. We regard both P and N_P as algebraic groups over F . Then $P = TN_P$. Let

$$K_{max} := K_\infty \times \prod_{v \text{ finite}} \mathrm{SL}_2(\mathcal{O}_v).$$

Then one has $G(\mathbb{A}) = P(\mathbb{A})K_{max}$. Let $K(\Gamma)_f \subseteq K_{max}$ be the compact subgroup of $G(\mathbb{A}_f)$ corresponding to Γ , i.e. $\Gamma = G(F) \cap K(\Gamma)_f$, where Γ and $G(F)$ are embedded diagonally into $G(\mathbb{A}_f)$. Let $g_{P_1}, \dots, g_{P_h} \in G(F)$ denote fixed representatives of $P(F) \backslash G(F) / \Gamma$. We assume $g_{P_1} = 1$. Let $P_i := g_{P_i}^{-1} P g_{P_i}$ denote the corresponding parabolic subgroups of G defined over F . In this section and the next one, we let N_{P_i} denote the unipotent radical of P_i regarded as an algebraic group over F and we shall denote by $N_{P_i, \infty}$ the corresponding real subgroup of $\mathrm{SL}_2(\mathbb{C})$. We embed $G(F)$, $P(F)$ as well as the elements g_{P_1}, \dots, g_{P_h} diagonally into $G(\mathbb{A})$. Then we have a canonical isomorphism

$$(5.1) \quad \mathcal{I}_{\mathbb{A}} : P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f \cong \bigsqcup_{i=1}^h (\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C})$$

which is defined as follows. By the strong approximation theorem one has $G(\mathbb{A}) = G(F) \mathrm{SL}_2(\mathbb{C}) K(\Gamma)_f$. This implies that each $g \in G(\mathbb{A})$ can be written as

$$(5.2) \quad g = b g_{P_i} g_\infty k_f,$$

where $b \in P(F)$, $g_{P_i} \in \{g_{P_1}, \dots, g_{P_h}\}$ is uniquely determined and g_∞ is unique up to $\Gamma \cap P_i(F) = \Gamma \cap N_{P_i, \infty}$. Let $\pi : G(\mathbb{A}) \rightarrow P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f$ denote the projection. Then according to (5.2), for $g \in G(\mathbb{A})$ we set $\mathcal{I}_{\mathbb{A}}(\pi(g)) := (\Gamma \cap N_{P_i, \infty}) g_\infty$, where $(\Gamma \cap N_{P_i, \infty}) g_\infty$ denotes the equivalence class of g_∞ in $(\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C})$. We let $\mathcal{I}_{\mathbb{A}, P_i} : P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f \rightarrow (\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C})$ be the maps induced by $\mathcal{I}_{\mathbb{A}}$. The map $\mathcal{I}_{\mathbb{A}}$ induces an isomorphism

$$(5.3) \quad (\mathcal{I}_{\mathbb{A}})^* : \bigoplus_{i=1}^{\kappa(\Gamma)} (C^\infty(N_{P_i, \infty} \backslash \mathrm{SL}_2(\mathbb{C})) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_\infty} \\ \xrightarrow{\sim} (C^\infty(P(F)N(\mathbb{A}) \backslash G(\mathbb{A}) / K(\Gamma)_f) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_\infty} =: W.$$

Here K_∞ acts on the C^∞ -spaces by right translation and on $(\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m)$ by $\mathrm{Ad}^* \otimes \rho(m)$. We shall denote this representation also by $\nu_1(\rho(m))$. We regard the real subgroups M_P and A_P of $\mathrm{SL}_2(\mathbb{C})$ from (3.1) or more generally the real subgroups M_{P_i} and A_{P_i} for a parabolic P_i introduced above as subgroups of $G(\mathbb{A})$. For $\sigma \in \hat{M}_P$ with $[\nu_1(\rho(m)) : \sigma] \neq 0$ and $\lambda \in \mathbb{C}$ we let $W_{\bar{\rho}_m}(\sigma, \lambda)$ be defined by:

$$(5.4) \quad W_{\bar{\rho}_m}(\sigma, \lambda) = \{f \in W : \forall g \in G(\mathbb{A}), a \in A_P, m \in M_P, f(amg) = \xi_{\lambda+1}(a) \sigma^{-1}(m) f(g)\}.$$

Let $K_{P_i, \infty} := g_{P_i}^{-1} K_{\infty} g_{P_i}$. Then $K_{P_i, \infty}$ acts on $(\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m)$ by conjugating with g_{P_i} . We let $\mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ be the space of all $f \in (C^\infty(N_{P_i, \infty} \setminus \mathrm{SL}_2(\mathbb{C})) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_{P_i, \infty}}$ which additionally satisfy

$$f(a_{P_i} m_{P_i} g) = \xi_{P_i, \lambda+1}(a_{P_i}) \sigma_{P_i}(m_{P_i}^{-1}) f(g) \quad \forall g \in \mathrm{SL}_2(\mathbb{C}), \quad \forall a_{P_i} \in A_{P_i}, \quad \forall m_{P_i} \in M_{P_i}.$$

Here $\xi_{P_i, \lambda+2}$ and σ_{P_i} are the characters which arise from ξ_λ and σ by conjugating with g_{P_i} . If ν is a finite-dimensional representation of K on a complex vector space V , we let $V^{\sigma_{P_i}}$ denote the σ_{P_i} -isotypical component of for the restriction of the representation ν to M_{P_i} . Then we have the following Lemma.

Lemma 5.1. *For $f \in W_{\bar{\rho}_m}(\sigma, \lambda)$ define $\mu_{\sigma, \lambda}(f) \in \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$ by*

$$\mu_{\sigma, \lambda}(f) := \sum_{i=1}^{\kappa(\Gamma)} (\mathcal{I}_{\mathbb{A}, P_i}^{-1})^* f(1) = \sum_{i=1}^{\kappa(\Gamma)} f(g_{P_i})$$

Then $\mu_{\sigma, \lambda}$ defines an isomorphism

$$\mu_{\sigma, \lambda} : W_{\bar{\rho}_m}(\sigma, \lambda) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$$

Proof. It is easy to see that each function $(\mathcal{I}_{\mathbb{A}, P_i}^{-1})^* f$ belongs to $\mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ and is therefore determined by its value at 1, which moreover belongs to $((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$. On the other hand, for $\Phi_{P_i} \in ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$ we define $\Phi_{P_i; \lambda} \in \mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ by

$$(5.5) \quad \Phi_{P_i; \lambda}(n_{P_i} a_{P_i} k) := \xi_{P_i, \lambda+1}(a_{P_i}) \nu_1(\rho_m)(k^{-1}) \Phi_{P_i}.$$

Let $\mu(\sigma, \lambda)^{-1}(\Phi_{P_i}) := \mathcal{I}_{\mathbb{A}}^*(\Phi_{P_i; \lambda})$. Then it immediately follows from the definitions that $\mu(\sigma, \lambda)$ and $\mu(\sigma, \lambda)^{-1}$ are inverse to each other. \square

We shall from now on identify $T(\mathbb{A})$ with the ring of ideles \mathbb{A}^* by sending $x \in \mathbb{A}^*$ to the diagonal matrix $\mathrm{diag}(x, x^{-1})$. Let $U(\Gamma)_f := T(\mathbb{A}) \cap K(\Gamma)_f$. Let $\sigma = \sigma_k$, $k \in \mathbb{Z}$ and let $\lambda \in \mathbb{C}$. Then we combine σ^{-1} and λ to a character $\chi_{\infty, \sigma, \lambda}$ of $MA = T_\infty \cong \mathbb{C}^*$ by putting

$$(5.6) \quad \chi_{\infty, \sigma, \lambda}(z) := |z|^{2(\lambda+1)} \left(\frac{\bar{z}}{|z|} \right)^k.$$

We let $\mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ denote the set of all Hecke characters $\chi : F^* \setminus \mathbb{A}^* \rightarrow \mathbb{C}$ which are trivial on $U(\Gamma)_f$ and which satisfy $\chi_\infty = \chi_{\infty, \sigma, \lambda}$. If $|\cdot|_{\mathbb{A}}$ denotes the norm on the adeles, then each $\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ can be uniquely written as

$$(5.7) \quad \chi = |\cdot|_{\mathbb{A}}^{2(\lambda+1)} \chi_1,$$

where χ_1 is unitary. For $\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ with χ as in (5.7) we define

$$w_0 \chi = \chi^{-1} = |\cdot|_{\mathbb{A}}^{2(-\lambda+1)} \bar{\chi}_1 \in \mathcal{H}(w_0 \sigma, -\lambda, K(\Gamma)_f).$$

Since $T(\mathbb{A})$ normalizes $N(\mathbb{A})$ the group $T(\mathbb{A})$ acts on $W_{\bar{\rho}_m}(\sigma, \lambda)$ by left translations and thus we obtain a decomposition of $W_{\bar{\rho}_m}(\sigma, \lambda)$ into χ -isotypical subspaces:

$$(5.8) \quad W_{\bar{\rho}_m}(\sigma, \lambda) = \bigoplus_{\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)} W_{\bar{\rho}_m}(\sigma, \lambda)_\chi.$$

Let $t_{P_1}, \dots, t_{P_h} \in T(\mathbb{A}_f)$ denote fixed representatives of $T(F) \backslash T(\mathbb{A}_f) / U(\Gamma)_f$. Then for $f \in W_{\bar{\rho}_m}(\sigma, \lambda)$, its projection f_χ onto $W_{\bar{\rho}_m}(\sigma, \lambda)_\chi$ is given by

$$(5.9) \quad f_\chi(g) = \frac{1}{\kappa(\Gamma)} \sum_{i=1}^{\kappa(\Gamma)} \bar{\chi}(t_{P_i}) f(t_{P_i} g).$$

Now we use the notations of the previous sections for the various cohomology groups. We can canonically identify the Lie algebra \mathfrak{n}_{P_i} of $N_{P_i, \infty}$ with \mathfrak{n}_P . Then we have canonical embeddings

$$\bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ \hookrightarrow \bigoplus_{i=1}^{\kappa(\Gamma)} (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m); \quad \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- \hookrightarrow \bigoplus_{i=1}^{\kappa(\Gamma)} (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m).$$

Moreover, an easy computation shows that in the present case these embeddings in fact give isomorphisms

$$\begin{aligned} \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ &\cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{m+2}}; \\ \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- &\cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{-m-2}}. \end{aligned}$$

Thus together with Lemma 5.1, we obtain isomorphisms

$$(5.10) \quad \mu_+(m) : W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ \cong H^1(\partial \bar{X}; \bar{V}(m))_+$$

and

$$(5.11) \quad \mu_-(m) : W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- \cong H^1(\partial \bar{X}; \bar{V}(m))_-.$$

We shall denote the operator from $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ to $W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ induced by $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ and the isomorphisms $\mu_\pm(m)$ by $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ too. Then it is well-known that for $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ and $g \in G(\mathbb{A})$ one has

$$\underline{\mathbf{C}}(\sigma_{-m-2}, m/2) f(g) = \int_{N(\mathbb{A})} f(w_0 n g) dn, \quad w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With respect to the decompositions of $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ resp. $W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ into Hecke-isotypical subspaces given in (5.8), the operator $\underline{C}(\sigma_{-m-2}, m/2)$ splits as

$$\underline{C}(\sigma_{-m-2}, m/2) = \bigoplus_{\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)} C(\chi),$$

where $C(\chi) : W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)_\chi \rightarrow W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)_{w_0\chi}$. Let $f \in W(\sigma_{-m-2}, m/2)_\chi$. For our later purposes we can assume that $f = f_\infty \otimes \bigotimes_{v \text{ finite}} f_v$. Then one has

$$C(\chi)f = J_\infty(\sigma_{-m-2}, m/2)f_\infty \otimes \bigotimes_{v \text{ finite}} C_v(\chi_v)f_v.$$

where the operators $C_v(\chi_v)$ are defined by

$$(5.12) \quad C_v(\chi_v)f_v = \int_{F_v} f_v \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) dx$$

The integral above can be computed explicitly, resulting in the next lemma. Let us set notation for it. Let χ_1 be as in (5.7). For v a finite place of F we let $\chi_{1,v}$ be the local component of χ_1 at v . Then we say that χ_1 is unramified at v if $\ker(\chi_{1,v}) \supseteq 1 + \mathcal{O}_v$. Otherwise we say that χ_1 is ramified at v . If χ_1 is unramified at v , then $\chi_{1,v}(\pi_v)$ does not depend on the choice of π_v (this happens in particular if $n_v = 0$). For v a finite place, χ_1 unramified at v and $s \in \mathbb{C}$ the local L -factor $L_v(\chi_{1,v}, s)$ is defined by

$$(5.13) \quad L_v(\chi_{1,v}, s) := \frac{1}{1 - \chi_{1,v}(\pi_v)|\pi_v|^{-s}}.$$

Then the following Lemma holds.

Lemma 5.2. *Let $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ and v be a finite place of F . Then:*

(i) *If $n_v = 0$, one has*

$$C_v(\chi_v)f_v(1_v) = \frac{L_v(\chi_{1,v}, m)}{L_v(\chi_{1,v}, m-1)}f_v(\text{Id}).$$

(ii) *If $n_v > 0$ and χ is ramified at v , then for $k_v \in K_{\max, v}$ one has*

$$C_v(\chi_v)f_v(k_v) = I_v(k_v)f_v(k_v),$$

where $I_v(k_v) \in |\pi_v|^{-2n_v m} \overline{\mathbb{Z}}$.

(iii) *If $n_v > 0$ and χ is unramified at v , then for $k_v \in K_{\max, v}$ one has*

$$C_v(\chi_v)f_v(k_v) = \frac{L_v(\chi_{1,v}, m)}{L_v(\chi_{1,v}, m-1)}I_v(k_v)f_v(k_v),$$

where $I_v(k_v) \in |\pi_v|^{-2n_v m} \overline{\mathbb{Z}}$.

Proof. We prove (i) first. For $x \in F_v, |x|_v > 1$ we have the Iwasawa decomposition

$$w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

On the other hand for $g = nak$ we have $f_v(g) = \alpha(a)^s \chi_{1,v}(a) f(\text{Id})$. Hence we get

$$\begin{aligned}
I_v &= \int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x)^{-1} f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx + \int_{\mathcal{O}_v} f_v \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx \\
&= \left(\int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x)^{-1} dx + \int_{\mathcal{O}_v} 1 dx \right) f_v(\text{Id}) \\
&= \left(\sum_{k \geq 1} (1 - q^{-1}) q^k \cdot \chi_{1,v}(\pi_v)^k q^{-2sk} + 1 \right) f_v(\text{Id}) \\
&= \left(1 + \frac{(\chi_{1,v}(\pi_v) q^{2s-1})^{-1} (1 - q^{-1})}{1 - \chi_{1,v}(\pi_v) q^{-2s+1}} \right) f_v(\text{Id}) = \frac{1 - \chi_{1,v}(\pi_v) q^{-2s}}{1 - \chi_{1,v}(\pi_v) q^{-2s+1}} \cdot f_v(\text{Id}).
\end{aligned}$$

Now let us prove (ii). It obviously suffices to deal with $k_v = \text{Id}$. Recall from the preceding proof that :

$$(5.14) \quad I_v = \int_{|x|_v > 1} |x|^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx + \int_{\mathcal{O}_v} f_v \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx.$$

The second term is a linear combination of integers (values of f_v) with coefficients in $\mathbb{Z}[q^{-n}]$ (the measure of a coset on which f_v is constant is equal to q^{-n}) and hence lies in $q^{-n} \overline{\mathbb{Z}}$. It remains to deal with the second term. Since χ_1 is ramified at v we have for any $k \in \mathbb{Z}$ the equality

$$\int_{|x|_v = q^k} \chi_{1,v}(x) dx = 0.$$

It follows that :

$$\begin{aligned}
\int_{|x|_v > 1} |x|^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx &= \int_{1 < |x|_v \leq q^n} |x|^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx \\
&= \sum_{a \in v\mathcal{O}_F/v^n} |a|_v^{2s-2} \chi_{1,v}(a) f_v \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} |a|^{-1} \int_{1+a^{-1}v^n\mathcal{O}_v} \chi_{1,v}(x) dx.
\end{aligned}$$

Now there are two possibilities for the integral on the second line: either $\chi_{1,v}$ is trivial on $1 + a^{-1}v^n$, in which case the integral equals $q^{-n}|a|^{-1}$, or the integral vanishes. In either case it lies in $q^{-n}\mathbb{Z}$, hence the sum lies in $q^{-(2s-1)n}\overline{\mathbb{Z}}$.

The proof of (iii) is just a combination of those of (i) and (ii). The decomposition (5.14) is still valid, and the estimate for the denominator of the first factor done there is still valid. For the first one we have :

$$\begin{aligned}
\int_{|x|_v < 1} |x|_v^{2-2s} \chi_{1,v}(x) f \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} dx &= \sum_{a \in \mathcal{O}_F/v^n} f_v \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \int_{a+v^n\mathcal{O}_v} |x|_v^{2-2s} \chi_{1,v}(x) dx \\
&= q^{-n} \sum_{a \neq 0} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \chi_{1,v}(a) |a|_v^{2s-2} + f_v(\text{Id}) \int_{|x|_v \leq q^{-n}} |x|_v \chi_{1,v}(x) dx
\end{aligned}$$

(since χ_1 is not ramified at v it is constant on every coset $a + v^n \mathcal{O}_v$). The terms with $a \neq 0$ belong to $q_v^{-(2s-1)n} \overline{\mathbb{Z}}$, and the same computation as in the proof of (i) yields that the last term lies in $q^{-2sn} \frac{L_v(\chi_1, 2s)}{L_v(\chi_1, 2s-1)} \overline{\mathbb{Z}}$. \square

Finally, the term $J_\infty(\sigma_{-m-2}, m/2)(f_\infty)$, which is always a ratio of Γ -functions, can be described explicitly in the present case. There is $\Phi \in ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^\sigma$ such that $f_\infty = \Phi_{m/2}$. Moreover, there is an M_∞ -equivariant isomorphism

$$\nu_1(\rho_m)(w_0) : ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^\sigma \cong ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^{w_0\sigma}.$$

The representation $\nu_1(\rho_m)$ of K is not irreducible. However, if ν_{m+2} denotes the representation of K of highest weight $m+2$ in the canonical parametrization, then ν_{m+2} occurs with multiplicity one in $\nu_1(\rho_m)$ and belong to the ν_{m+2} -isotypical subspace. Thus we have

$$(5.15) \quad J_\infty(\sigma_{-m-2}, m/2)(\Phi_{m/2}) = c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) \cdot (\nu_1(\rho_m)(w_0)\Phi)_{-m/2},$$

where $c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) \in \mathbb{C}$ is the value of generalized Harish-Chandra c-function. In the present case, the latter is known explicitly. Namely, by [Coh74, Appendix 2], taking the different parametrizations into account, one has

$$(5.16) \quad c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) = \frac{1}{\pi} \frac{1}{im + m + 2}.$$

6. ESTIMATION OF THE DENOMINATOR OF THE C-MATRIX

We keep the notation of the preceding section. Our goal here is to prove the following estimate for the denominator of the intertwining matrices.

Proposition 6.1. *Let Γ be a (principal) congruence subgroup of Γ_D . Then there exists a constant $C_0(\Gamma)$ such that one can estimate*

$$\log |d_{Eis, \mathbf{C}}(\Gamma, \bar{\rho}_m)| \leq C_0(\Gamma) m \log(m)$$

for all $m \in \mathbb{N}$.

Using the maps $\mu_\pm(m)$ from (5.10) and (5.11) we obtain distinguished integral lattices $\mu_+^{-1}(m)(H^1(\partial \overline{X}; \bar{\Lambda}(m))_+)$ in the space $W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ and $\mu_-^{-1}(m)(H^1(\partial \overline{X}; \bar{\Lambda}(m))_-)$ in $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$. More generally, if $R \subset \mathbb{C}$ is a ring with $\mathbb{Z} \subset R$ we will say that $f \in W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ is defined over R if it is in the image of $\mu_+^{-1}(m)(H^1(\partial \overline{X}; \bar{\Lambda}(m))_+ \otimes_{\mathbb{Z}} R)$ and we make the corresponding definition for $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$. The decomposition of $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ with respect to Hecke characters given in (5.8) does not respect the \mathbb{Z} -structure on this space just introduced. In other words, if $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ is defined over \mathbb{Z} and if we decompose

$$(6.1) \quad f = \sum_{\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)} f_\chi,$$

then we cannot expect the f_χ to be defined over \mathbb{Z} . However, we have the following Proposition which controls this defect.

Proposition 6.2. *There exists an algebraic integer $\alpha \in \overline{\mathbb{Z}}$ which depends on the group Γ but not on the representation $\rho(m)$ such that if $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ is defined over \mathbb{Z} then $\alpha^m f_\chi$ is defined over $\overline{\mathbb{Z}}$ for each f_χ in the decomposition (6.1).*

Proof. The character $\chi_{\infty, \sigma_{-m-2}, m/2}$ is the character

$$(6.2) \quad \chi_{\infty, m+2} : T(\mathbb{C}) \rightarrow \mathbb{C}, \quad \chi_{\infty, m+2} \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) := z^{m+2}$$

on $T(\mathbb{C})$. Let $\mathcal{H}^1(\mathfrak{n}_P; \bar{\Lambda}(m))_-$ denote the integral lattice in corresponding to $H^1(\partial \bar{X}; \bar{\Lambda}(m))_-$. Without loss of generality, we may assume that $f = \mu_-^{-1}(\Phi)$, where $\Phi \in H^1(\mathfrak{n}_P; \bar{\Lambda}(m))_-$. We fix $\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)$. We have

$$\mu_- \left((\mu_-^{-1}(\Phi))_\chi \right) = \sum_{i=1}^{\kappa(\Gamma)} (\mu_-^{-1}(\Phi))_\chi(g_{P_i}).$$

On the other hand, by (5.9), for each i we have

$$(6.3) \quad (\mu_-^{-1}(\Phi))_\chi(g_{P_i}) = \frac{1}{\kappa(\Gamma)} \sum_{j=1}^{\kappa(\Gamma)} \chi(t_{P_j})(\mu_-^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i})$$

For each i, j there exists a unique $l = l(i, j) \in \{1, \dots, h\}$ and a $g_\infty(i, j) \in \mathrm{SL}_2(\mathbb{C})$ such that

$$(6.4) \quad t_{P_j}^{-1}g_{P_i} = bg_{P_l}g_\infty(i, j)k,$$

where $b \in P(F)$, $k \in K_f(\Gamma)$. We fix $g_\infty(i, j)$ satisfying (6.4). If $g_{P_l} \neq g_{P_l} = 1$, then, by definition one has $(\mu_-^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i}) = 0$. On the other hand, if $g_{P_l} = 1$, then by definition one has

$$(6.5) \quad (\mu_-^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i}) = \Phi_{m/2}(g_\infty(i, j)),$$

where $\Phi_{m/2} = \Phi_{P, m/2} \in \mathcal{E}_P(\sigma_{-m-2}, m/2, \nu_1(\rho_m))$ is as in (5.5). Let $g_\infty(i, j) = p_\infty(i, j)k_\infty(i, j)$, where $p_\infty(i, j) \in P_\infty$, $k_\infty(i, j) \in K_\infty$. Then:

$$\Phi_{m/2}(g_\infty(i, j)) = \nu_1(\rho_m)(k_\infty(i, j))^{-1} \Phi_{m/2}(p_\infty(i, j)).$$

One has $\Phi_{m/2}(p_\infty(i, j)) \in (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{-m-2}}$ and $\nu_1(\rho_m)(k_\infty(i, j))^{-1} \Phi_{m/2}(p_\infty(i, j)) \in (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{-m-2}}$. It is easy to see that this implies $k_\infty(i, j) \in M_\infty$, i.e. $g_\infty \in P_\infty$. Moreover, together with (6.4) it follows that $g_\infty \in P(F)$. Thus one can write

$$g_\infty(i, j) = t(g_\infty(i, j))n(g_\infty(i, j)),$$

$t(g_\infty(i, j)) \in T(F)$, $n(g_\infty(i, j)) \in N(F)$. We write $t(g_\infty(i, j)) = \mathrm{diag}(t_{i,j}, t_{i,j}^{-1})$ with $t_{i,j} \in F^*$. Then by (6.2) and by the definition of $\Phi_{m/2}$ we have $\Phi_{m/2}(g_\infty(i, j)) = t_{i,j}^{m+2} \Phi$. Thus if $\alpha_{i,j} \in \mathcal{O}_F^*$ is the denominator of $t_{i,j}$, i.e. $\alpha_{i,j} t_{i,j} \in \mathcal{O}_F^*$, it follows that

$$(6.6) \quad \alpha_{i,j}^{m+2} \Phi_{m/2}(g_\infty(i, j)) \in H^1(\mathfrak{n}_P, \bar{\Lambda}(m))_- \otimes_{\mathbb{Z}} \mathcal{O}_F.$$

Next, each Hecke character $\chi : F^* \backslash \mathbb{A}^* \rightarrow \mathbb{C}$ which is trivial on $U(\Gamma)$ and which satisfies $\chi_\infty = \chi_{m+2, \infty}$ is of the form $\tilde{\chi}^{m+2}$, where $\tilde{\chi} : F^* \backslash \mathbb{A}^* \rightarrow \mathbb{C}$ is a character which is trivial on

$U(\Gamma)$ and satisfies $\tilde{\chi}_\infty(\text{diag}(z, z^{-1})) = z$ for $z \in \mathbb{C}^*$. The set of such characters $\tilde{\chi}$ is finite. Thus it follows that there exists a $\beta \in \overline{\mathbb{Z}}$ such that:

$$(6.7) \quad \beta^{m+2} \chi(t_{P_j}) \in \overline{\mathbb{Z}}$$

for all $j = 1, \dots, \kappa(\Gamma)$ and all $\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)$. Combining (6.3), (6.5), (6.6) and (6.7) we get the statement in the proposition. \square

For a unitary Hecke character χ_1 we consider the Hecke L-function

$$L(\chi_1, s) = \prod_v L_v(\chi_1, v, s)$$

Recall that for a place v where χ does not ramify we defined the local factor L_v in (5.13); we take the convention that $L_v(\chi_1, s) := 1$ if χ_1 is ramified at v . The infinite product converges absolutely for $\text{Re}(s) > 1$ and admits a meromorphic continuation to \mathbb{C} . For $a \in \overline{\mathbb{Q}}$ we shall denote by $|a|_{\overline{\mathbb{Q}}/\mathbb{Q}}$ or simply $|a|$ its absolute norm given by

$$|a|_{\overline{\mathbb{Q}}/\mathbb{Q}} := |B|, \quad B = \prod_{b \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot a} b \in \mathbb{Q},$$

where $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of \mathbb{Q} . The following proposition evaluates the denominator of the quotient of L -values appearing in the computation of the intertwining operators in Lemma 5.2. It is essentially contained in the work of Damerell [Dam70], [Dam71]: our argument consists in keeping track of norm estimates along the steps in the proof of the main theorem of [Dam70].

Proposition 6.3. *Let \mathfrak{J} be an ideal in \mathcal{O}_D . There is $A \in \mathbb{Z}_{>0}$ such that for all unitary Hecke characters χ whose conductor divides \mathfrak{J} and whose infinite part $\chi_\infty(z) = (z/|z|)^n = (\overline{z}/z)^{n/2}$ for an even integer n , and for any integer $s \in \{0, \dots, n/2\}$ there is an integer $a' \in \overline{\mathbb{Z}}$ such that $|a'| \leq (n!)^A$ and we have :*

$$(6.8) \quad 2^{n-2s+1} a' \frac{L(\chi, s)}{L(\chi, s-1)} \in \pi \overline{\mathbb{Z}}.$$

Proof. We first indicate how the arguments used in [Dam70] yield the following result, which is a more precise version of Theorem 1 in loc. cit.

Lemma 6.4. *There is an $\Omega \in \mathbb{C}^\times$ (depending only on the field F) such that the following holds. Let n be an even integer, χ a unitary Hecke character of F with infinite part $\chi_\infty(z) = (\overline{z}/z)^{n/2}$ and s an integer in the range $\{0, \dots, n/2\}$. Then*

$$(6.9) \quad \pi^{n/2-s} L(\chi, s) / \Omega^n$$

is an algebraic number, whose degree over \mathbb{Q} is bounded by a constant depending only on F and the conductor \mathfrak{f} of χ and whose absolute norm is bounded by $C(n!)^A$ for positive integers C, A depending only on F .

Proof. In the proof of this lemma all our numbered references are to Damerell's paper [Dam70]. Damerell's statement includes only the algebraicity, but his arguments give the full statement above as we shall now explain. Formula (6.2) yields the following expression for the normalized L -value occuring above :

$$(\dagger) \quad \pi^{n/2-s} L(\chi, s) / \Omega^n = M \cdot \sum_{i=1}^h \hat{\mathfrak{A}}_i \chi_f(\hat{\mathfrak{A}}_i)^{-1} \sum_{\beta \in \mathfrak{A}_i / \mathfrak{B}_i} \chi_f(\beta) \psi^p F_n(\beta \Omega, s, \Omega \mathfrak{B}_i)$$

where:

- M is an algebraic number depending on \mathfrak{f} and F ;
- $\mathfrak{A}_1, \dots, \mathfrak{A}_h$ are integral ideals representing the elements in the class-group of F ;
- $\mathfrak{B}_i = \mathfrak{f} \mathfrak{A}_i$;
- ψ is a number depending on F and $p = n/2 - s$;
- F_n is a particular function which we will analyze below ;
- $\Omega \in \mathbb{C}^\times$ is a particular number depending only on F .

The next step is Lemma 5.2, which yields an expression of $F_n(\cdot, \cdot, \Lambda)$ in terms of arithmetic invariants of the elliptic curve \mathbb{C}/Λ . More precisely, let \wp be the Weierstrass function associated to this elliptic curve, and let $s > 1$. Then we have :

$$(*) \quad \psi^p F_n(z, s, \Lambda) = \sum_{t+u+v=p} \frac{p!}{t!u!v!} h(z)^t (-\wp)^u (-1)^v K_{q-u}^v(z).$$

where :

- $h(z)$ is a rational fraction (with coefficients in \mathbb{Q}) in $\wp(kz), \wp'(kz), \wp''(kz)$ where $k = 1, \dots, \ell - 2$, with ℓ the exponent of the finite abelian group $\mathfrak{A}_i / \mathfrak{B}_i$ (see Lemma 4.3) ;
- K_j^i is a polynomial (with coefficients in \mathbb{Q}) in $h, \wp(z), \wp'(z), g_2(\Lambda)$ and $g_3(\Lambda)$ and \wp (Corollary 4.1) ;
- φ is a constant depending on the curve \mathbb{C}/Λ .

The key fact is then that all points at which the various \wp occuring in (\dagger) are estimated are of bounded finite order on the elliptic curves: this yields algebraic equations for the relevant values of \wp and its derivative whose degree is bounded (depending on F and \mathfrak{f}). The values of g_2 and g_3 are algebraic of degree depending on the elliptic curve (this is where the choice of x enters, see the remark after Lemma 2.1). It then follows that the values of \wp'' are algebraic of bounded degree, as we can see by differentiating in z the equation

$$(d\wp/dz)^2 = 4\wp^3 - g_2\wp - g_3$$

satisfied by \wp , which yields

$$(b) \quad d^2\wp/dz^2 = 6\wp^2 - g_2/2$$

(cf. (3.10), (3.11)). All of this proves that the factors $h(z)$ and K_{q-u}^v in $(*)$ are algebraic of bounded (depending on F, \mathfrak{f}) degree. It remains to deal with φ . Choosing a $\tau \in \mathcal{O}_F$,

equation (6.1) yields the expression

$$\varphi = (\tau\bar{\tau} - \tau^2)^{-1} \sum_{\substack{\rho \in \tau b\mathfrak{B}_i/x\mathfrak{B}_i \\ \rho \neq 0}} \wp(\rho/\tau)$$

which is algebraic of bounded degree. This finishes the proof that all factors of the summands in (*) are algebraic of bounded degree. To finish the proof that the normalized L -value itself is so we need only note that since the character χ_f is of bounded finite order (depending on \mathfrak{f}), its values in (\dagger) are roots of unity of bounded degree. Thus all terms in (\dagger) are algebraic integers of bounded degree.

Now we must bound the absolute norm of the right-hand side in (\dagger). It is obvious from (\dagger), (*) and the proof that the degree is bounded that it suffices to prove that the valuation of K_i^j , for $2 \leq i+2 \leq j \leq n$ is bounded by $C(n!)^A$ for some constant C . To do this we must return to the arguments of Damerell; in the proof of Lemma 3.3 he shows that for $j \geq 2$ one has

$$K_j^0(z) = (-1)^j \frac{d^{j-2}\wp(z)}{dz^{j-2}}.$$

From this an easy recursive argument using the identity (b) allows to prove that for $j \geq 2$ $K_j^0(z)$ is a polynomial in $\wp(z)$, $\wp'(z)/2$ and $g_2/12$ of degree less than $2j$ in each variable, with coefficients in \mathbb{Z} that are $\ll (2j)! \cdot N^j$ for some integer $N \in \mathbb{Z}_{>0}$; we will denote this polynomial by $P_j^0 \in \mathbb{Z}[X_1, \dots, X_4]$ (the last variable represents $g_3/4$).

Damerell proves that for $0 \leq i < j$ there is a polynomial $P_j^i \in \mathbb{Z}[X_1, \dots, X_4]$ such that $K_j^i(z) = P_j^i(\wp(z), \wp'(z)/2, g_2/12, g_3/4)$. For this he uses the recurrence relation (3.12), which is :

$$(\#) \quad K_{j+1}^{i+1}(z) = i \frac{\wp'(z)}{2} \cdot \frac{1}{j} K_j^{i-1}(z) - i \frac{g_2}{12} \cdot \frac{1}{j} K_{j-1}^{i-1}(z) - \frac{1}{j} D K_i^j(z)$$

where D is a differential operator (in both second variables of \wp). It is given explicitly for \wp, \wp', g_2 and g_3 in the equalities (3.7), which we rewrite here :

$$\begin{aligned} Dg_2 &= -6g_3, & Dg_3 &= -\frac{1}{3}g_2^2, \\ D\wp &= -2\wp^2 - \frac{g_2}{3}, & D\wp' &= -3\wp \cdot \wp'. \end{aligned}$$

Together with (#) these finally yield that the degree of P_j^i in each variable is less than $2(i+j)$ and the coefficients are majorized by $2(i+j)!N^{(i+j)}$ for some $N \in \mathbb{Z}_{>0}$. It follows that for the values of z occuring in (\dagger) we have $|K_{q-u}^v(z)| \ll n!N^n$ at each place, hence the absolute norm is bounded by $(n!N^n)^{[E:\mathbb{Q}]}$. This finishes the proof of our statement. \square

Bounds for the denominators of special values of L -functions are also given by the work of Damerell. We will use the following statement to evaluate denominators of the L -part of the intertwining integrals: let n be an even integer, χ be a unitary Hecke character of F , with infinite part

$$\chi_\infty(z) = (\bar{a}/a)^{n/2}$$

and finite part χ_f of conductor \mathfrak{J} . Then [Dam71, Theorem 2] states that for the complex number $\Omega \in \mathbb{C}^\times$ appearing in Lemma 6.4 there is an algebraic integer $a \in \overline{\mathbb{Z}}$, whose absolute norm $|a| = |a|_{\overline{\mathbb{Q}}/\mathbb{Q}}$ is bounded independently of s, χ_∞ , such that for all integers $s \in \{0, \dots, n/2\}$ we have

$$(6.10) \quad 2^{n/2-s} a \cdot L(\chi, s) \in \Omega^n / \pi^{n/2-s} \overline{\mathbb{Z}}.$$

The proposition follows from this and the bound for the absolute norm of normalized L -values given in Lemma 6.4 (the transcendental factors cancel between the numerator and denominator). \square

We can finally put everything together to estimate the denominators of the \mathbb{C} -matrix.

Proof of Proposition 6.1. Let X_\pm be the integral vectors in \mathfrak{n}_P of weight ± 2 for \mathfrak{su}_2 and v_\pm the vectors of weight $\pm m$ for \mathfrak{su}_2 in $\bar{V}(m)$. Then a cohomology class ω_\pm in $\mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_\pm$ is integral if and only if $\int_{c_\pm} \omega \in \mathbb{Z}$ where c_\pm is the 1-cycle on $\Gamma_{P_\infty} \backslash \mathbb{H}^3$ with coefficients in $\bar{V}(m)$ associated to X_\pm and v_\pm . Moreover we have

$$\int_{c_\pm} \omega_\pm = \mu_\pm(m)^{-1}(\omega)(\text{Id}), \quad \int_{c_\pm} \underline{\mathbf{C}}(\omega_\pm) = \mathbf{C}(\mu_\mp(m)^{-1}(\omega))(\text{Id}).$$

Likewise a cohomology class $\omega_\pm \in \mathcal{H}^1(\mathfrak{n}_{P_i}; \bar{V}(m))_\pm$ with coefficients in $\bar{V}(m)$ is integral if and only if the value of a function $f = \mu_\pm^{-1}(\omega_\pm)$ at g_{P_i} is integral (up to an at most exponential factor in m coming from the non-integrality of the cycle associated to $\text{ad}(g_{P_i}) \cdot X_\pm$ and $\bar{\rho}_m(g_{P_i}) \cdot v_\pm$), and we have the same equivariance property viz. the operators $\underline{\mathbf{C}}$ and \mathbf{C} .

The functions $\mu_\pm(m)^{-1}(\omega)(g_{p_i})$ on $G(\mathbb{A})$ are K'_f -invariant where K'_f is the compact-open subgroup $\bigcap_{i=1}^h g_{P_i} K_f(\Gamma) g_{P_i}^{-1}$ of $G(\mathbb{A})$. So the statement in the proposition reduces to the following claim: let K'_f be a compact-open subgroup in $G(\mathbb{A}_f)$ and $f \in W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ corresponding to a rational integral cohomology class (the latter being defined as above, with $K_f(\Gamma)$ replaced by K'_f). Then we claim that there are $N, C, A \in \mathbb{Z}_{>0}$ depending only on K'_f such that we have

$$\mathbf{C}(s(m))f(\text{Id}) \in C^{-1} N^{-m} (m!)^A \mathbb{Z} f(\text{Id}).$$

To prove we note that it suffices to prove a similar result over $\overline{\mathbb{Z}}$, namely that for all f as above corresponding to a cohomology class with coefficients in $\bar{\Lambda}(m) \otimes \overline{\mathbb{Z}}$ we have

$$(6.11) \quad \mathbf{C}(s(m))f(\text{Id}) \in a^{-1} \mathbb{Z} f(\text{Id})$$

for an algebraic integer a with $|a|_{\overline{\mathbb{Q}}/\mathbb{Q}} \leq C(m!)^A$ (indeed, since we know a priori that if the right-hand side is defined over \mathbb{Q} the proposition follows by taking the product of Galois conjugates of each side).

Let us prove (6.11). First, it follows from Proposition 6.2 that it suffices to prove it for $f \in W_\chi$. By Lemma 5.2 and (5.15), (5.16) we get that it suffices to prove that

$$\frac{1}{\pi} \cdot \frac{L(\chi, m)}{L(\chi, m-1)} \in b^{-1} \overline{\mathbb{Z}}$$

for an $a \in \overline{\mathbb{Z}}$ with absolute norm $|b| \leq (m!)^A$. This last statement follows from Proposition 6.3. \square

7. BOUNDING THE TORSION FROM BELOW

In this section we prove the estimate (1.2) (and also (1.4), which we actually need to prove the former) from our main Theorem A.

7.1. Torsion in $H^*(\Gamma, \bar{\Lambda}(m))$. We first show directly that the order of the group $H^1(\Gamma; \bar{\Lambda}(m))_{tors}$ grows slower in m than our leading term. Since we work with a split algebraic group the proof of this is simpler than that of the corresponding statement in [MM13].

Lemma 7.1. *Let Γ be a congruence subgroup of Γ_D . Then*

$$\log |H^1(\Gamma; \bar{\Lambda}(m))_{tors}| = O(m \log m),$$

as $m \rightarrow \infty$.

Proof. One has $H^1(\Gamma; \bar{\Lambda}(m))_{tors} \cong H_0(\Gamma; \bar{\Lambda}(m))_{tors}$ by the universal coefficient theorem. Let $\Lambda^0(m)$ denote the submodule of $\Lambda(m)$ generated by all $(\rho_m(\gamma) - \text{Id})v$, where $v \in \Lambda(m)$ and $\gamma \in \Gamma$. Then $H_0(\Lambda(m)) = \Lambda(m)/\Lambda^0(m)$. There exists an $a \in \mathbb{N}$ such that for $n_a := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\bar{n}_a := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ one has $n_a, \bar{n}_a \in \Gamma$. If we let X, Y denote the standard basis of \mathbb{C}^2 , then $X^m, X^{m-1}Y, \dots, Y^m$ is a basis of $\Lambda(m)$ and in this basis, $\rho_m(n_a) - \text{Id}$ is represented by an upper triangular nilpotent matrix. For $j > i$, the entry in the i -th row, j -th column of this matrix is given by $a^{j-i} \binom{j-1}{j-i}$. Thus it follows inductively that $(l+1)aX^{m-l}Y^l \in \Lambda^0(m)$ for $0 \leq l < m$. On the other hand, one has $(\rho_m(\bar{n}_a) - \text{Id})XY^{m-1} = aY^m$. Thus one has $|H_0(\Lambda(m))| \leq a^{m+1}m!$. i.e. $\log |H_0(\Lambda(m))| = O(m \log m)$ as $m \rightarrow \infty$. For $\check{\Lambda}(m)$ one can argue similarly. \square

The main result we prove here is the exponential growth of the torsion subgroup of $H^2(\Gamma, \bar{\Lambda}(m))$; in the remainder of this subsection we will show how all the work done in sections 4–6 implies the following result.

Proposition 7.2. *We have*

$$(7.1) \quad \liminf_{m \rightarrow +\infty} \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|}{m^2} \geq \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right)$$

Proof. We first remark that it follows easily from the interpretation of $H^1(\partial\bar{X}; \bar{\Lambda}(m))_{\pm}$ in terms of closed $E_{\bar{\rho}_m}$ -valued holomorphic respectively antiholomorphic forms on the boundary $\partial\bar{X}$ that

$$(7.2) \quad [H^1(\partial\bar{X}; \bar{\Lambda}(m))_{free} : H^1(\partial\bar{X}; \bar{\Lambda}(m))_{-} \oplus H^1(\partial\bar{X}; \bar{\Lambda}(m))_{+}] = O(m \log m),$$

as $m \rightarrow \infty$.

Let \mathcal{A} and $C_1(\Gamma)$ be as in the end of section 3. Let \mathfrak{a} be a non-zero ideal of \mathcal{O}_D with $N(\mathfrak{a}) > C_1(\Gamma)$ and let $\mathfrak{a}_0 \in \mathcal{A}$ such that $n_{l,\Gamma(\mathfrak{a})} = n_{l,\Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. For brevity we shall use the following notation in the remaining computations:

$$R^i(X, \bar{\mathcal{L}}(m)) = \text{vol}_{\mathcal{B}^i(\Gamma, \bar{\rho}_m)}(H^i(\Gamma, \bar{\Lambda}(m))).$$

With this notation (4.4) becomes

$$(7.3) \quad \begin{aligned} \log \tau_{Eis}(X; E_{\bar{\rho}_m}) &= \log R^2(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m)) + \log |H^1(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}| \\ &\quad - \log R^1(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m)) - \log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|. \end{aligned}$$

By Lemmas 4.2 and 7.1 we have that

$$(7.4) \quad |\log (|H^1(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|)|, |\log R^2(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m))| \ll m \log(m).$$

On the other hand, by Lemma 4.1 together with Proposition 6.1 and (7.2) we have that

$$\liminf_{m \rightarrow +\infty} \frac{\log R^1(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m))}{m^2} \leq \liminf_{m \rightarrow +\infty} \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|}{m^2}$$

From (7.4) we get that in the expression (7.3) for the Reidemeister torsion $\tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})$ all terms but for $\log |H^2|$ and $\log R^1$ are $O(m \log(m))$ and using the preceding inequality we get that

$$(7.5) \quad \liminf_{m \rightarrow +\infty} \left(2 \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|}{m^2} \right) \geq \liminf_{m \rightarrow +\infty} \left(- \frac{\log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})}{m^2} \right).$$

On the other hand, we also get from (7.2) and Lemma 4.1 that

$$\liminf_{m \rightarrow +\infty} \left(\frac{-\log R^1(X_{\mathfrak{a}_0}, \bar{\mathcal{L}}(m))}{m^2} \right) \leq 0$$

from which and (7.4) (used for \mathfrak{a}_0 instead of \mathfrak{a}) it follows that

$$(7.6) \quad \liminf_{m \rightarrow +\infty} \left(\frac{\log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}_m})}{m^2} \right) \leq 0.$$

Putting together (7.5) and (7.6) we get that

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \left(2 \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a}_0)} \cdot \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|}{m^2} \right) &\geq \\ \liminf_{m \rightarrow +\infty} \left(\frac{-[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a}_0)} \cdot \frac{\log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})}{m^2} + \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a})} \cdot \frac{\log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}_m})}{m^2} \right). \end{aligned}$$

Finally, the right-hand side above converges to

$$\begin{aligned} 2 \cdot \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)] \cdot [\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*| \pi} \left(\frac{1}{N(\mathfrak{a}_0)} - \frac{1}{N(\mathfrak{a})} \right) \text{vol}(\Gamma_D \backslash \mathbb{H}^3) \\ = 2 \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*| N(\mathfrak{a}_0)} \cdot \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \cdot \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right) \end{aligned}$$

by Proposition 3.2 (which we are allowed to use for the representation $\rho_m \oplus \rho_m$ instead of ρ_m since the analytic or Reidemeister torsion of the former is the square of that of the latter). This finishes the proof of Proposition 7.2. \square

7.2. Independance from the lattice. Here we prove that $\log |H^2(\Gamma, \Lambda_m)_{tors}|$ does not depend on the choice of lattices $\Lambda_m \subset V(m)$ up to an error term of size $m \log(m)$.

Proposition 7.3. *Let Γ be a finite-index subgroup of the Bianchi group Γ_D . There is a constant C depending only on Γ such that for any $m \geq 1$ and any two Γ -invariant lattices Λ_1, Λ_2 in $V(m)$ we have*

$$\left| \log \left(\frac{|H^2(\Gamma, \Lambda_1)_{tors}|}{|H^2(\Gamma, \Lambda_2)_{tors}|} \right) \right| \leq Cm \log(m).$$

We will deduce the proposition from the two next lemmas.

Lemma 7.4. *Let Γ be a subgroup of a Bianchi group, ρ be a representation of $\mathrm{SL}_2(\mathbb{C})$ on a vector space V and Λ, Λ' two $\rho(\Gamma)$ -invariant lattices in V such that $M \cdot \Lambda \subset \Lambda' \subset \Lambda$ for some integer $M \in \mathbb{Z}_{>0}$. Let $\mathcal{L}, \mathcal{L}'$ be the local systems on $X = \Gamma \backslash \mathbb{H}^3$ induced by Λ, Λ' and E_ρ the Euclidean bundle on X induced by ρ . Then we have*

$$1 \leq \frac{R^1(X, \mathcal{L}')}{R^1(X, \mathcal{L})} \leq M^{\dim H^1(X, E_\rho)}.$$

Proof. Let $h = \dim H^1(X, E_\rho)$ and let $c_1, \dots, c_h \in Z^1(X, \mathcal{L})$ such that the cohomology classes $[c_1], \dots, [c_h]$ generate the free part of $H^1(X, \mathcal{L})$. Then each $M \cdot c_i$ belongs to $Z^1(X, \mathcal{L}')$ and together the $M \cdot [c_i]$ generate a finite-index subgroup of $H^1(X, \mathcal{L}')$. Thus we get

$$M \cdot H^1(X, \mathcal{L}) \subset H^1(X, \mathcal{L}')$$

and the inequality

$$R^1(X, \mathcal{L}') \leq [H^1(X, \mathcal{L}) : M \cdot H^1(X, \mathcal{L})] R^1(X, \mathcal{L}) = M^h R^1(X, \mathcal{L})$$

follows immediately. \square

Lemma 7.5. *There is a constant $c \in \mathbb{Z}_{>0}$ depending on Γ such that if Λ_1, Λ_2 are two Γ -invariant lattices in $V(m)$ then there exists $a \in \mathbb{Q}$ such that*

$$a\Lambda_1 \subset \Lambda_2 \subset a(m!)^{-c}\Lambda_1.$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the pairing on $V(m) = \mathrm{Sym}^m \mathbb{C}^2$ induced by the determinant on $\mathbb{C}^2 \times \mathbb{C}^2$ (which is hence nondegenerate and Γ -invariant) and let Λ'_1 be the $\langle \cdot, \cdot \rangle$ -dual lattice of Λ_1 in $V(m)$, that is

$$\Lambda'_1 = \{v \in V(m) : \forall u \in \Lambda_1, \langle u, v \rangle \in \mathbb{Z}\}.$$

Then

$$(7.7) \quad m!\Lambda'_1 \subset \Lambda_1 \subset (m!)^{-1}\Lambda'_1$$

as follows from the expression of $\langle \cdot, \cdot \rangle$ in coordinates (see for example [Ber08, 2.4]). Now let u be a primitive vector in Λ_2 which is a vector of maximal weight for the standard

parabolic subgroup of $\mathrm{SL}_2(\mathbb{C})$ in $V(m)$ (i.e. a rational multiple of X^m); there exists an $a \in \mathbb{Q}$ such that au is a primitive vector in Λ'_1 . Then $\Lambda_3 := \langle \Gamma \cdot au \rangle \subset \Lambda'_1$: indeed, for any $v \in \Lambda_1$ and $\gamma \in \Gamma$ we have

$$\langle v, \gamma \cdot au \rangle = \langle \gamma^{-1} \cdot v, au \rangle \in \mathbb{Z}.$$

From this and (7.7) we get that $\Lambda_3 \subset (m!)^{-1}\Lambda_1$. By arguments similar to those used in the proof of Lemma 7.1 (which we will detail after we explain how to conclude the proof from there), we also have that

$$(7.8) \quad a\Lambda_2 \subset N^{-m}(m!)^{-2}\Lambda_3$$

for some $N \in \mathbb{Z}_{>0}$ depending on Γ . It finally follows that

$$a\Lambda_2 \subset N^{-m}(m!)^{-3}\Lambda_1$$

which proves half the lemma; the second half also follows by a completely symmetric argument.

Let us explain how (7.8) is proved. For ease of notation we will suppose that $au = X^m$. We put $\Lambda_0 = \Lambda(m) = \mathcal{O}_D X^m \oplus \mathcal{O}_D X^{m-1}Y \oplus \dots \oplus \mathcal{O}_D Y^m$. Let $z \in \mathbb{Z}$ such that $n_z \in \Gamma$ and $k \in \{1, \dots, m\}$; suppose that $bX^{m-k}Y^k \in \Lambda_2$ for some $b \in F$. Then we get that

$$\Lambda_2 \ni n_z^k \cdot bX^{m-k}Y^k = bk!z^k X^m$$

so that $bk!z^m \in \mathcal{O}_D$. Thus we get that $\Lambda_2 \subset (z^m m!)^{-1}\Lambda_0$. On the other hand the proof of Lemma 7.1 yields that $[\Lambda_0 : \Lambda_3] \leq z^m m!$, hence $\Lambda_2 \subset (z^m m!)^{-2}\Lambda_3$. \square

Proof of Proposition 7.3. Let \mathcal{L}_i be the local system on X induced by the lattice Λ_i . We have by (7.3) that

$$\frac{R^2(X, \mathcal{L}_1) \cdot |H^1(\Gamma, \Lambda_1)_{tors}|}{R^1(X, \mathcal{L}_1) \cdot |H^2(\Gamma, \Lambda_1)_{tors}|} = \tau_{Eis}(X, \rho_m) = \frac{R^2(X, \mathcal{L}_2) \cdot |H^1(\Gamma, \Lambda_2)_{tors}|}{R^1(X, \mathcal{L}_2) \cdot |H^2(\Gamma, \Lambda_2)_{tors}|}.$$

By Lemmas 4.2 and 7.1 we have that H^1, R^2 are $\ll m \log(m)$ for whichever lattice, and by Lemmas 7.4 and 7.5 we get that

$$\left| \log \left(\frac{R^1(X, \mathcal{L}_1)}{R^1(X, \mathcal{L}_2)} \right) \right| \ll m \log(m)$$

and we can thus conclude that the remaining terms $\log(H^2(\Gamma, \Lambda_i))$ in the Reidemeister torsion differ by at most $Cm \log(m)$ for some $C > 0$ depending on Γ . \square

7.3. Conclusion. Let $\bar{\Lambda}'(m)$ be the lattice $\Lambda(m) \oplus \Lambda(m)$ in $V(m) \oplus V(m)$. By Propositions 7.3 and 7.2 we get that

$$\liminf \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}'(m))_{tors}|}{m^2} \geq \frac{\mathrm{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right).$$

Since $H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}'(m)) \cong H^2(\Gamma(\mathfrak{a}), \Lambda(m))^2$ we get (1.2). The estimate (1.4) is proven exactly as in Lemma 7.1.

8. BOUNDING THE TORSION FROM ABOVE

In this section we proof equation (1.3) from Theorem A : we give the proof for $\bar{\Lambda}(m)$ -coefficients, the case of $\Lambda(m)$ follows immediately by Proposition 7.3. The main ingredient is the following lemma, usually attributed to O. Gabber and C. Soulé (we note that it is also an important tool in V. Emery's a priori bound for the torsion in the homology of certain arithmetic lattices, see [Eme14]). We refer the reader to [Sau14, Lemma 3.2] for a proof.

Lemma 8.1. *Let $A := \mathbb{Z}^a$ with standard basis $(e_i)_{i=1,\dots,a}$ and let $B := \mathbb{Z}^b$. Equip $B \otimes_{\mathbb{Z}} \mathbb{R}$ with the Euclidian norm $\|\cdot\|$. Let $\phi : A \rightarrow B$ be \mathbb{Z} -linear and assume that there exists $\alpha \in \mathbb{R}$ such that $\|\phi(e_i)\| \leq \alpha$ for each $i = 1, \dots, a$. Then one has*

$$|\mathrm{coker}(\phi)_{\mathrm{tors}}| \leq \alpha^{\min(a,b)}$$

Next, we have the following elementary Lemma.

Lemma 8.2. *Let $\{v_j := e_1^{m-i} e_2^j : i = 0, \dots, m\}$, denote the standard integral basis of the lattice $\Lambda_m \subset V(m)$. Equip $V(m)$ with the inner product such that the v_i form an orthonormal basis and let $\|\cdot\|_{\mathrm{End}(V_m)}$ denote the corresponding norm on $\mathrm{End}(V_m)$. Then for each $\gamma \in \Gamma_D$ one has*

$$\|\rho_m(\gamma)\|_{\mathrm{End}(V_m)} \leq \|\rho_1(\gamma)\|_{\mathrm{End}(V_1)}^m$$

Proof. This follows immediately from the definitions. \square

Now we can estimate the torsion from above as follows.

Proposition 8.3. *Let Γ be a finite index, torsion-free subgroup of Γ_D . Then there exists a constant c_Γ such that one can estimate*

$$\log |H^2(\Gamma, \bar{\Lambda}(m))_{\mathrm{tors}}| \leq c_\Gamma m^2$$

for each $m \in \mathbb{N}$.

Proof. Let $X := \Gamma \backslash \mathbb{H}^3$. Let \mathcal{K} be a smooth triangulation of X and let $\tilde{\mathcal{K}}$ denote its lift to a smooth triangulation of \mathbb{H}^3 . For each q let $C_q(\mathcal{K}) := \{\sigma_{1,q}, \dots, \sigma_{N(\Gamma,q),q}\}$ denote the simplicial q -chains of \mathcal{K} , where $N(\Gamma, q) \in \mathbb{N}$ depends on \mathcal{K} . Let $C_q(\tilde{\mathcal{K}})$ denote the simplicial q -chains of $\tilde{\mathcal{K}}$ and let $\tilde{\partial}_q : C_q(\tilde{\mathcal{K}}) \rightarrow C_{q-1}(\tilde{\mathcal{K}})$ be the corresponding boundary operator. For each $\sigma_{i,q}$ we fix a $\tilde{\sigma}_{i,q} \in \tilde{\mathcal{K}}$ such that $\pi_*(\tilde{\sigma}_{i,q}) = \sigma_{i,q}$, where $\pi : \tilde{X} \rightarrow X$ is the covering map. The group Γ acts on $C_q(\mathcal{K})$ and we denote the corresponding action simply by \cdot . For each $\sigma_{i,q}$ there exist elements $\gamma_{k,q-1} \in \Gamma$, $k = 1, \dots, N(q-1, \Gamma)$, such that

$$\tilde{\partial}_q(\tilde{\sigma}_{i,q}) = \sum_{k=1}^{N(q-1, \Gamma)} \gamma_{k,q-1} \cdot \tilde{\sigma}_{k,q-1}.$$

Let $C_q(\mathcal{K}; \Lambda_m) := C_q(\mathcal{K}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_m$. Then the homology groups $H_*(\Gamma; \Lambda_m)$ are isomorphic to the homology groups $H_*(C_*(\mathcal{K}; \Lambda_m))$ of the complex

$$(C_*(\mathcal{K}; \Lambda_m), \partial_{*,\rho_m}) := (C_*(\mathcal{K}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_m, \tilde{\partial}_* \otimes \mathrm{Id}).$$

Let $\{v_0, \dots, v_m\}$ denote the standard integral basis of Λ_m as in Lemma 8.2. Then an integral basis of $C_q(\mathcal{K}; \Lambda_m)$ is given by

$$B_q(\mathcal{K}; \Lambda_m) := \{\tilde{\sigma}_{i,q} \otimes v_j : i = 1, \dots, N(\Gamma, q) : j = 0, \dots, m\}.$$

We equip $C_q(\mathcal{K}; \Lambda_m) \otimes_{\mathbb{Z}} \mathbb{R}$ with the inner product for which $B_q(\mathcal{K}; \Lambda_m)$ is an orthonormal basis and denote the corresponding norm by $\|\cdot\|_{C_q(\mathcal{K}; V(m))}$. Then we have

$$\partial_{q;\rho_m}(\tilde{\sigma}_{i,q} \otimes v_j) = \sum_{k=1}^{N(q-1,\Gamma)} \tilde{\sigma}_{k,q-1} \otimes (\rho_m(\gamma_{k,q-1}^{-1})v_j)$$

and thus by the definition of the norms we have

$$\begin{aligned} \|\partial_{q;\rho_m}(\tilde{\sigma}_{i,q} \otimes v_j)\|_{C_{q-1}(\mathcal{K}; V(m))} &\leq N(q-1, \Gamma) \max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_m(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_m)} \\ (8.1) \quad &= N(q-1, \Gamma) \left(\max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_1(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_1)} \right)^m, \end{aligned}$$

where the last step follows from Lemma 8.2. We put

$$c_0(\Gamma) := \max_q \max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_1(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_1)}.$$

Then $c_0(\Gamma)$ depends on Γ and the triangulation \mathcal{K} , but not on the local system Λ_m . If we apply Lemma 8.1 with $A := C_q(\mathcal{K}; \Lambda_m)$, $B := C_{q-1}(\mathcal{K}; \Lambda_m)$, $\phi := \partial_{q;\rho_m}$ and $\alpha := N(\Gamma, q-1)c_0(\Gamma)^m$, then, using that $\text{rk}_{\mathbb{Z}} A = (m+1)N(q, \Gamma)$, $\text{rk}_{\mathbb{Z}} B = (m+1)N(q-1, \Gamma)$ we obtain from (8.1) that

$$|(coker(\partial_{q;\rho_m}))_{tors}| \leq (N(\Gamma, q-1)c_0(\Gamma)^m)^{(m+1) \min\{N(q, \Gamma), N(q-1, \Gamma)\}}$$

For $\check{\Lambda}_m$ one argues in the same way. Thus the proposition follows by applying the universal coefficient theorem. \square

Remark 1. Using the KAK -decomposition, it should be possible to generalize Lemma 8.2 and thus the proof of Proposition 8.3 to arithmetic subgroups Γ of arbitrary connected semisimple Liegroups G defined over \mathbb{Q} which satisfy $\delta(G) = 1$. For suitable rays $\rho_{\lambda}(m)$ of \mathbb{Q} -rational representations of G of highest weight $m\lambda$ with Γ -invariant integral lattices $\Lambda(\rho_{\lambda}(m))$, this should give an upper bound of the corresponding sizes of all twisted cohomological torsion subgroups $H_{tors}^*(\Gamma, \Lambda(\rho_{\lambda}(m)))$ by $C(\Gamma)m \dim \rho_{\lambda}(m)$. Such a bound can be regarded as complementary to the lower bound obtained in the compact case in [MP14b].

Let $\mathfrak{a}_0 \in \mathcal{A}$ be as in the previous section. If we apply Proposition 8.3 for the group $\Gamma(\mathfrak{a}_0)$ instead of the group $\Gamma(\mathfrak{a})$ and also use Proposition 3.2, we can improve the constant in the upper bound of the size of $m^{-2} \log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|$ and thus prove (1.3). Namely,

arguing similar as in the proof of (1.2) given in the previous section, we obtain

$$\begin{aligned}
& \limsup_{m \rightarrow \infty} m^{-2} \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{\#(\mathcal{O}_D^*)N(\mathfrak{a}_0)} \log |H_{tors}^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))| - \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})} c(\Gamma_0) \\
& \leq \limsup_{m \rightarrow \infty} m^{-2} \left(-\frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{\#(\mathcal{O}_D^*)N(\mathfrak{a}_0)} \log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}(m)}) + \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{\#(\mathcal{O}_D^*)N(\mathfrak{a})} \log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}(m)}) \right) \\
& \quad + \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{\#(\mathcal{O}_D^*)N(\mathfrak{a}_0)} c(\Gamma(\mathfrak{a}_0)).
\end{aligned}$$

Invoking Proposition 3.2, equation (1.3) follows.

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